

QUANTUM DIMENSIONS AND FUSION RULES OF THE VOA $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$

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ABSTRACT. In this article, we determine quantum dimensions and fusion rules for the orbifold code VOA $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$. As an application, we also construct certain 3-local subgroups inside the automorphism group of the VOA V^\sharp , where V^\sharp is a holomorphic VOA obtained by the \mathbb{Z}_3 -orbifold construction on the Leech lattice VOA.

1. INTRODUCTION

The study of vertex operator algebras as modules of Virasoro VOA was first initiated by Dong, Mason and Zhu [DMZ94]. They proved that the Moonshine VOA V^\natural contains 48 mutually orthogonal elements such that each of them will generate a copy of the rational Virasoro VOA $L(\frac{1}{2}, 0)$ inside V^\natural and the sum of these 48 conformal vectors is the Virasoro element of V^\natural . This discovery turns out to be very important for the study of the Moonshine VOA [Don94, Miy04]. It also leads to the development of the theory of framed vertex operator algebras (cf. [Miy98, Miy04] and [DGH98]). Roughly speaking, a framed VOA is a simple VOA which contains a full sub VOA $F \cong L(\frac{1}{2}, 0)^{\otimes n}$ such that $\text{rank}(V) = \text{rank}(F) = n/2$. There are many interesting examples, which include the famous Moonshine VOA. Moreover, it is known that if V is a framed VOA with the weight one subspace $V_1 = 0$, then the full automorphism group $\text{Aut}(V)$ is finite [Miy04, GL12]. Therefore, the theory of framed VOAs is very useful for studying certain finite groups such as the Monster.

In [DMZ94], the Virasoro VOA $L(\frac{1}{2}, 0)$ was constructed inside the lattice type VOA $V_{\mathbb{Z}\alpha}^+$, where $\langle \alpha, \alpha \rangle = 4$. In fact, $V_{\mathbb{Z}\alpha}^+ \cong V_{\sqrt{2}A_1}^+ \cong L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$. Therefore, a framed VOA with integral central charge k may also be considered as an extension of the tensor product of the orbifold VOA $V_{\sqrt{2}A_1}^+$. In this article, we consider a generalization of framed VOAs. Namely, we replace the VOA $V_{\sqrt{2}A_1}^+$ by another orbifold VOA $(V_{\sqrt{2}A_2}^\tau)$, where τ is a lift of a fixed point free isometry of order 3 in $O(\sqrt{2}A_2)$, and study certain extensions of the VOA $(V_{\sqrt{2}A_2}^\tau)^{\otimes n}$. We first study a certain integral lattice $L_{\mathcal{C} \times \mathcal{D}}$ that are constructed from an \mathbb{F}_4 -code \mathcal{C} and an \mathbb{F}_3 -code \mathcal{D} as an extension of the lattice $(\sqrt{2}A_2)^{\oplus n}$. We also study the irreducible modules for the orbifold VOA $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$. As our main result, we determine the quantum dimensions and the fusion rules for all irreducible $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ -modules. In particular, we show that all irreducible $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ -modules are simple current modules if the \mathbb{F}_4 -code

\mathcal{C} is self-dual. Moreover, the fusion ring for $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ is isomorphic to a group ring of an elementary abelian 3-group and the set of all inequivalent irreducible $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ -modules forms a quadratic space over \mathbb{F}_3 if \mathcal{C} is self-dual.

As an application, we study the case when \mathcal{C} is isomorphic to the Hexacode in detail. In this case, $L_{\mathcal{C} \times \{0\}}$ is isomorphic to the Coxeter-Todd lattice K_{12} of rank 12. We show that the full automorphism group of $V_{K_{12}}^\tau$ is isomorphic to $\Omega_8^-(3).2$. Several 3-local subgroups of the VOA V^\sharp , obtained from \mathbb{Z}_3 -orbifold construction using the Leech lattice VOA, are also studied and computed explicitly.

This article is organized as follows. In Section 2, we review some basic properties of the VOA $V_{\sqrt{2}A_2}^\tau$ and the notion of quantum dimensions. In Section 3, we review a construction of the integral lattice $L_{\mathcal{C} \times \mathcal{D}}$ from some \mathbb{F}_4 and \mathbb{F}_3 -codes. Some basic facts about the lattice VOA $V_{L_{\mathcal{C} \times \mathcal{D}}}$ and its \mathbb{Z}_3 -orbifold $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ will also be recalled. In Section 4, we compute the quantum dimensions of the orbifold VOA $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$. In Section 5, we compute the fusion rules among irreducible $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ -modules. As an application, we construct in Section 6 certain 3-local subgroups inside the automorphism group of the VOA V^\sharp , where V^\sharp is a holomorphic VOA obtained by a \mathbb{Z}_3 -orbifold construction on the Leech lattice VOA.

2. PRELIMINARIES AND BASIC PROPERTIES

The VOAs $V_{\sqrt{2}A_2}$ and $V_{\sqrt{2}A_2}^\tau$. In this section we review some facts about the orbifold VOA $V_{\sqrt{2}A_2}^\tau$. For general background concerning lattice VOA, we refer to [FLM88, LL03].

Let α_1, α_2 be the simple roots of type A_2 and set $\alpha_0 = -(\alpha_1 + \alpha_2)$. Then $\langle \alpha_i, \alpha_i \rangle = 2$ and $\langle \alpha_i, \alpha_j \rangle = -1$ if $i \neq j$, $i, j \in \{0, 1, 2\}$. Set $\beta_i = \sqrt{2}\alpha_i$ and let $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ be the lattice spanned by β_1 and β_2 . Then L is isometric to $\sqrt{2}A_2$.

Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ denote the Galois field of four elements, where ω is a root of $x^2 + x + 1 = 0$ over \mathbb{F}_2 . We adopt the similar notation as in [KLY03, DLT⁺04] and denote the cosets of L in the dual lattice $L^\perp = \{\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid \langle \alpha, L \rangle \subset \mathbb{Z}\}$, as follows:

$$\begin{aligned} L^0 &= L, & L^1 &= \frac{-\beta_1 + \beta_2}{3} + L, & L^2 &= \frac{\beta_1 - \beta_2}{3} + L, \\ L_0 &= L, & L_1 &= \frac{\beta_2}{2} + L, & L_\omega &= \frac{\beta_0}{2} + L, & L_{\bar{\omega}} &= \frac{\beta_1}{2} + L, \end{aligned} \tag{2-1}$$

and

$$L^{(i,j)} = L_i + L^j,$$

for $i = 0, 1, \omega, \bar{\omega}$ and $j = 0, 1, 2$. Then, $L^{(i,j)}$, $i \in \mathbb{F}_4, j \in \mathbb{Z}_3 = \{0, 1, 2\}$ are all the cosets of L in L^\perp and $L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. It is shown in [Don93] that there are exactly 12 isomorphism classes of irreducible V_L -modules, which are given by $V_{L^{(i,j)}}$, $i = 0, 1, \omega, \bar{\omega}$ and $j = 0, 1, 2$.

Consider the isometry $\tau : L \rightarrow L$ defined by

$$\beta_1 \mapsto \beta_2 \mapsto \beta_0 \mapsto \beta_1.$$

Then τ is fixed point free of order three and can be lifted naturally to an automorphism of V_L by mapping

$$a^1(-n_1) \cdot a^k(-n_k)e^b \mapsto (\tau a^1)(-n_1) \cdot (\tau a^k)(-n_k)e^{\tau b}.$$

By abuse of notation, we also use τ to denote the lift.

In [KLY03], it was shown that there are exactly three irreducible τ -twisted V_L -modules and three irreducible τ^2 -twisted V_L -modules, up to isomorphism. They are denoted by $V_L^{T,j}(\tau)$ or $V_L^{T,j}(\tau^2)$ for $j = 0, 1, 2$.

The automorphism τ acts on the set of inequivalent irreducible V_L -modules by $V_{L^{(i,j)}} \circ \tau$. Note also that $V_{L^{(i,j)}} \circ \tau \cong V_{L^{(\bar{\omega}i,j)}}$. We denote

$$U[\varepsilon] = \{u \in U \mid \tau u = \exp(2\pi\sqrt{-1}\varepsilon/3)u\},$$

for any τ -invariant V_L -module U and $\varepsilon = 0, 1, 2$. Irreducible modules for the orbifold VOA V_L^τ are classified by Tanabe and Yamada [TY07] and the following result was proved.

Proposition 2.1. [TY07] *The VOA V_L^τ is a simple, rational, C_2 -cofinite, and of CFT type. There are exactly 30 inequivalent irreducible V_L^τ -modules. They are represented as follows.*

- (i) $V_{L^{(0,j)}}[\varepsilon]$ for $j, \varepsilon = 0, 1, 2$.
- (ii) $V_{L^{(\omega,j)}}$ for $j = 0, 1, 2$.
- (iii) $V_L^{T,j}(\tau^i)[\varepsilon]$ for $i = 1, 2$ and $j, \varepsilon = 0, 1, 2$.

Weights of these modules are given by given by (see Tanabe and Yamada[TY07, (5,10)]):

$$\begin{aligned} \text{wt } V_{L^{0,j}}[\varepsilon] &\in \frac{2j^2}{3} + \mathbb{Z}, \\ \text{wt } V_L^{T,j}(\tau^i)[\varepsilon] &\in \frac{10 - 3(j^2 + \varepsilon)}{9} + \mathbb{Z}, \end{aligned}$$

for $i = 1, 2, j, \varepsilon \in \mathbb{Z}_3$.

Quantum Dimension. We now review the notion of quantum dimension introduced by Dong et al. [DJX13].

Let V be a VOA of central charge c and let $M = \bigoplus_{n \in \mathbb{Z}_+} M_{\lambda+n}$ be a V -module, where λ is the lowest conformal weight of M . The *normalized character* of M is defined as

$$\text{ch } M(q) := q^{\lambda-c/24} \sum_{n \in \mathbb{Z}_+} \dim M_{\lambda+n} q^n,$$

where $q = e^{2\pi\sqrt{-1}z}$ and $z = x + \sqrt{-1}y$ is in the complex upper half-plane \mathbb{H} .

The following notion of quantum dimension is introduced by Dong et al. [DJX13].

Definition 2.2. Suppose $\text{ch } V(q)$ and $\text{ch } M(q)$ exist. The *quantum dimension of M over V* is defined as

$$\text{qdim}_V M := \lim_{y \rightarrow 0^+} \frac{\text{ch } M(\sqrt{-1}y)}{\text{ch } V(\sqrt{-1}y)}, \quad (2-2)$$

where y is a positive real number.

From now on, we will omit the variable q and write the character $\text{ch } M(q)$ as $\text{ch } M$ instead. Fundamental properties of quantum dimension are also proved in their paper.

Proposition 2.3. [DJX13] *Let V be a simple, rational, C_2 -cofinite VOA of CFT-type and $V \cong V'$. Let W, W^1, W^2 be V -modules. Then*

- (i) $\text{qdim}_V W \geq 1$.
- (ii) qdim_V is multiplicative, that is $\text{qdim}_V(W^1 \times W^2) = \text{qdim}_V W^1 \cdot \text{qdim}_V W^2$, where $W^1 \times W^2$ denotes the fusion product.
- (iii) A V -module W is a simple current if and only if $\text{qdim}_V W^1 = 1$.
- (iv) $\text{qdim}_V W = \text{qdim}_V W'$, where W' is the contragredient dual of W .

Remark 2.4. Recall that a simple V -module M is a *simple current* module if and only if for every simple V -module W , $M \times W$ exists and is also a simple V -module.

Quantum dimensions of irreducible V_L^τ -modules are computed in [Che13].

Proposition 2.5. [Che13] *We have*

- (i) $\text{qdim}_{V_L^\tau} V_{L^{(0,j)}}[\varepsilon] = 1$ for $j, \varepsilon = 0, 1, 2$.
- (ii) $\text{qdim}_{V_L^\tau} V_{L^{(\omega,j)}} = 3$ for $j = 0, 1, 2$.
- (iii) $\text{qdim}_{V_L^\tau} V_L^{T,j}(\tau^i)[\varepsilon] = 2$ for $i = 1, 2$ and $j, \varepsilon = 0, 1, 2$.

3. THE VOAS $V_{L_{\mathcal{C} \times \mathcal{D}}}$ AND $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$

\mathbb{Z}_3 and \mathbb{F}_4 -codes. We first review the coding theory concerned in this paper. All codes mentioned in this paper are linear codes. From now on, we fix $\ell \in \mathbb{N}$.

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_\ell)$ be a codeword of length ℓ , its *support* is defined to be $\text{Supp}(\boldsymbol{\lambda}) = \{i \mid \lambda_i \neq 0\}$. The cardinality of $\text{Supp}(\boldsymbol{\lambda})$, denoted by $\text{wt}(\boldsymbol{\lambda})$, is called the (*Hamming*) *weight* of $\boldsymbol{\lambda}$. A code \mathcal{S} is said to be *even* if $\text{wt}(\boldsymbol{\lambda})$ is even for every $\boldsymbol{\lambda} \in \mathcal{S}$. Let \mathcal{S} be a code of length ℓ . The (*Hamming*) *weight enumerator* of \mathcal{S} is defined to be

$$W_{\mathcal{S}}(X, Y) = \sum_{\boldsymbol{\lambda} \in \mathcal{S}} X^{\ell - \text{wt}(\boldsymbol{\lambda})} Y^{\text{wt}(\boldsymbol{\lambda})}, \quad (3-1)$$

which is a homogeneous polynomial of degree ℓ .

We consider the inner products for codes over \mathbb{F}_4 and \mathbb{Z}_3 as follows. For codes over \mathbb{F}_4 , we use the Hermitian inner product, i.e.,

$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{\ell} x_i \bar{y}_i$$

for $\mathbf{x} = (x_1, \dots, x_\ell), \mathbf{y} = (y_1, \dots, y_\ell) \in \mathbb{F}_4^\ell$, where $\bar{x} = x^2$ is the *conjugate* of $x \in \mathbb{F}_4$. For \mathbb{Z}_3 -codes, we use the usual Euclidean inner product:

$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{\ell} x_i y_i \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{Z}_3^\ell.$$

Let $\mathcal{K} = \mathbb{F}_4$ or \mathbb{Z}_3 . For a \mathcal{K} -code \mathbf{S} of length ℓ with inner product given as above, we define its dual code by

$$\mathbf{S}^\perp = \{\boldsymbol{\lambda} \in \mathcal{K}^\ell \mid \boldsymbol{\lambda} \cdot \boldsymbol{\mu} = 0 \text{ for all } \boldsymbol{\mu} \in \mathbf{S}\}.$$

A \mathcal{K} -code \mathbf{S} is said to be *self-orthogonal* if $\mathbf{S} \subset \mathbf{S}^\perp$ and *self-dual* if $\mathbf{S} = \mathbf{S}^\perp$.

Remark 3.1. By [HP03, Thm.1.4.10], an \mathbb{F}_4 -code \mathbf{C} is even if and only if \mathbf{C} is Hermitian self-orthogonal. Note that the underlying “additive” group structure of \mathbb{F}_4 is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, an even \mathbb{F}_4 -code \mathbf{C} is also an even “additive” $\mathbb{Z}_2 \times \mathbb{Z}_2$ code. Moreover, \mathbf{C} is τ -invariant since it is \mathbb{F}_4 -linear. In Tanabe and Yamada [TY07], even τ -invariant $\mathbb{Z}_2 \times \mathbb{Z}_2$ codes are used. Instead of the Hermitian inner product, they used the trace Hermitian inner product defined by $\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{\ell} x_i \bar{y}_i + \bar{x}_i y_i$.

In the notation of [RS98], our code \mathbf{C} belongs to the family 4^H , while Tanabe and Yamada’s code belongs to the family 4^{H+} . If the code \mathbf{C} is also linear, then its dual \mathbf{C}^\perp in 4^{H+} coincides with the dual of \mathbf{C} in 4^H . Therefore, these two notions are essentially the same and almost all theorems we proved in this paper have analogous statements in their setting.

The lattice $L_{\mathbf{C} \times \mathbf{D}}$ and the VOAs $V_{L_{\mathbf{C} \times \mathbf{D}}}$ and $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$. In this paper, we use a boldface lowercase letter \mathbf{x} to denote a vector or a sequence of length ℓ and its i -th coordinate is denoted by x_i . That is

$$\mathbf{x} = (x_1, \dots, x_\ell).$$

From now on, we let \mathbf{C} be a self-orthogonal \mathbb{F}_4 -code of length ℓ and let \mathbf{D} be a self-orthogonal \mathbb{Z}_3 -code of the same length. First we review a construction of an even lattice from \mathbf{C} and \mathbf{D} [KLY03, TY13].

For $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{F}_4^\ell$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_\ell) \in \mathbb{Z}_3^\ell$, we define

$$L_{\boldsymbol{\lambda} \times \boldsymbol{\delta}} := \{(x_1, \dots, x_\ell) \in (L^\perp)^{\oplus \ell} \mid x_i \in L^{(\lambda_i, \delta_i)}, i = 1, \dots, \ell\}.$$

For subsets $\mathbf{P} \subset \mathbb{F}_4^\ell$ and $\mathbf{Q} \subset \mathbb{Z}_3^\ell$, we define

$$L_{\mathbf{P} \times \mathbf{Q}} := \bigcup_{\boldsymbol{\lambda} \in \mathbf{P}, \boldsymbol{\delta} \in \mathbf{Q}} L_{\boldsymbol{\lambda} \times \boldsymbol{\delta}} \subset (L^\perp)^{\oplus \ell}.$$

Let τ acts diagonally on $(L^\perp)^{\oplus \ell}$ and hence it induces an action on $V_{(L^\perp)^{\oplus \ell}}$. The purpose of this paper is to determine quantum dimensions and fusion rules of irreducible $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ -modules.

Proposition 3.2 ([TY13]). *Let \mathcal{C} be a self-orthogonal \mathbb{F}_4 -code of length ℓ and \mathcal{D} be a self-orthogonal \mathbb{Z}_3 -code of the same length. Then the subset $L_{\mathcal{C} \times \mathcal{D}}$ is an even sublattice of $(L^\perp)^{\oplus \ell}$. Moreover, the dual lattice $(L_{\mathcal{C} \times \mathcal{D}})^\perp = L_{\mathcal{C}^\perp \times \mathcal{D}^\perp}$.*

Proposition 3.3. [Don93, DL96, TY13] *Let \mathcal{C} be a self-orthogonal \mathbb{F}_4 -code of length ℓ and \mathcal{D} be a self-orthogonal \mathbb{Z}_3 -code of the same length. Let $V_{L_{\mathcal{C} \times \mathcal{D}}}$ be the lattice VOA associated to $L_{\mathcal{C} \times \mathcal{D}}$. Then we have the following.*

(i) *The set of all inequivalent irreducible $V_{L_{\mathcal{C} \times \mathcal{D}}}$ -modules is given by*

$$\{V_{L_{(\lambda+\mathcal{C}) \times (\delta+\mathcal{D})}} \mid \lambda + \mathcal{C} \in \mathcal{C}^\perp / \mathcal{C}, \delta + \mathcal{D} \in \mathcal{D}^\perp / \mathcal{D}\}.$$

(ii) *We have $V_{L_{(\lambda+\mathcal{C}) \times (\delta+\mathcal{D})}} \circ \tau \cong V_{L_{(\tau^{-1}(\lambda)+\mathcal{C}) \times (\delta+\mathcal{D})}}$.*

(iii) *For $i = 1, 2$, there are exactly $|\mathcal{D}^\perp / \mathcal{D}|$ inequivalent irreducible τ^i -twisted $V_{L_{\mathcal{C} \times \mathcal{D}}}$ -modules. They are represented by $(V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta}(\tau^i), Y^{\tau^i})$ for $\eta \in \mathcal{D}^\perp \bmod \mathcal{D}$.*

Remark 3.4. As τ -twisted $V_{L \oplus L}$ -modules,

$$V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta}(\tau) \cong \bigoplus_{\gamma \in \mathcal{D}} V_{L \oplus L}^{T, \eta - \gamma}(\tau),$$

for $i = 1, 2$. Furthermore, we have the following decomposition of $V_{L \oplus L}^{T, \eta}(\tau)$ into a direct sum of irreducible $(V_L^\tau)^{\otimes \ell}$ -modules.

$$V_{L \oplus L}^{T, \eta}(\tau) \cong \bigoplus_{(\varepsilon_1, \dots, \varepsilon_\ell) \in \mathbb{Z}_3^\ell} V_L^{T, \eta_1}(\tau)[\varepsilon_1] \otimes \cdots \otimes V_L^{T, \eta_\ell}(\tau)[\varepsilon_\ell]. \quad (3-2)$$

Similar for τ^2 -twisted modules.

Since τ acts trivially on \mathcal{D} ,

$$V_{L_{(\lambda+\mathcal{C}) \times (\delta+\mathcal{D})}} \cong V_{L_{(\lambda'+\mathcal{C}) \times (\delta'+\mathcal{D})}}$$

if and only if (1) $\lambda + \mathcal{C}$ and $\lambda' + \mathcal{C}$ belong to the same τ -orbit of \mathcal{C}^\perp ; and (2) $\delta + \mathcal{D} = \delta' + \mathcal{D}$ in $\mathcal{D}^\perp / \mathcal{D}$. Let $\mathcal{C}_{\equiv \tau}^\perp$ denote the set of all τ -orbits in \mathcal{C}^\perp . Then

$$\{V_{L_{\mathcal{C} \times (\delta+\mathcal{D})}}[\varepsilon], V_{L_{(\lambda+\mathcal{C}) \times (\delta+\mathcal{D})}} \mid \mathbf{0} \neq \lambda + \mathcal{C} \in \mathcal{C}_{\equiv \tau}^\perp \bmod \mathcal{C}, \delta + \mathcal{D} \in \mathcal{D}^\perp / \mathcal{D}\}$$

is a set of inequivalent irreducible $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ -modules, which are obtained from the irreducible (untwisted) $V_{L_{\mathcal{C} \times \mathcal{D}}}$ -modules.

It is usually very difficult to classify all irreducible modules of an orbifold VOA. Recently, Miyamoto gave a classification in the \mathbb{Z}_3 -orbifold case.

Proposition 3.5. [Miy13, Thm.A] *Let V be a simple VOA of CFT-type. Assume $V \cong V'$, its contragredient dual, and all V -modules are completely reducible. If σ is an automorphism of V of order three and a fixed point subVOA V^σ is C_2 -cofinite, then all V^σ -modules are completely reducible. Moreover, every simple V^σ -module appears as a V^σ -submodule of some σ^j -twisted (or ordinary) V -module.*

By this proposition, we can classify irreducible $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ -modules.

Proposition 3.6. *An irreducible $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ -module must belong to one of the following types.*

$$(i) V_{L_{\mathbf{C} \times (\delta + \mathbf{D})}}[\varepsilon], \quad (ii) V_{L_{(\lambda + \mathbf{C}) \times (\delta + \mathbf{D})}}, \quad (iii) V_{L_{\mathbf{C} \times \mathbf{D}}}^{T, \eta}(\tau^i)[\varepsilon],$$

where $i = 1, 2$, $\varepsilon \in \mathbb{Z}_3$, $\mathbf{0} \neq \lambda + \mathbf{C} \in \mathbf{C}_{\equiv \tau}^\perp \bmod \mathbf{C}$, $\eta \in \mathbf{D}^\perp(\bmod \mathbf{D})$ and $\delta + \mathbf{D} \in \mathbf{D}^\perp / \mathbf{D}$. In particular, there is no modules of the second type (ii) if \mathbf{C} is self-dual.

In this paper, our calculations depend heavily on the decomposition of the irreducible $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ -modules as $(V_L^\tau)^{\otimes \ell}$ -modules.

Proposition 3.7 ([KLY03, TY13]). *As modules of $(V_L^\tau)^{\otimes \ell}$, we have the following decomposition.*

$$(i) \quad V_{L_{\mathbf{C} \times (\delta + \mathbf{D})}}[\varepsilon] \cong V_{L_{\mathbf{0} \times (\delta + \mathbf{D})}}[\varepsilon] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathbf{C}_{\equiv \tau}} V_{L_{\gamma \times (\delta + \mathbf{D})}}; \quad (3-3a)$$

$$(ii) \quad V_{L_{\mathbf{C} \times \mathbf{D}}}^{T, \eta}(\tau^i)[\varepsilon] \cong \bigoplus_{\delta \in \mathbf{D}} \bigoplus_{e_1 + \dots + e_\ell \equiv \varepsilon \bmod 3} V_L^{T, \eta_1 - i\delta_1}(\tau^i)[e_1] \otimes \dots \otimes V_L^{T, \eta_\ell - i\delta_\ell}(\tau^i)[e_\ell], \quad (3-3b)$$

where $\delta = (\delta_1, \dots, \delta_\ell) \in \mathbb{Z}_3^\ell$ and $\mathbf{C}_{\equiv \tau}$ denote the set of all τ -orbits in \mathbf{C} .

In order to using properties about quantum dimensions, we need the following proposition.

Proposition 3.8. *The VOAs $V_{L_{\mathbf{C} \times \mathbf{D}}}$ and $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ are simple, rational, C_2 -cofinite VOAs of CFT type and are isomorphic to their contragredient dual, respectively.*

Assertions about lattice VOA is well-known. The simplicity and rationality of $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ are proved in [TY13]. The C_2 -cofiniteness of $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ is given in [Miy13, Thm B]. By [Li94, Cor.3.2], $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ is self-dual.

4. QUANTUM DIMENSIONS OF IRREDUCIBLE $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ -MODULES

In this section, we compute the quantum dimensions of irreducible $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ -modules. We will first consider the case when $\mathbf{D} = \{\mathbf{0}\}$ is the trivial \mathbb{Z}_3 -code. Results in this case are summarized in Theorem 4.10. The general case will be considered in Section 4 (see Theorem 4.11). In addition, we verify one conjecture about global dimensions proposed by Dong et. al. for the VOA $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau$ in this section.

Weight enumerators. For $\varepsilon = 0, 1, 2$, let

$$S_\varepsilon := \{\mathbf{x} := (x_1, \dots, x_\ell) \in \mathbb{Z}_3^\ell \mid \sum x_i \equiv \varepsilon \bmod 3\}.$$

We define the *weight enumerator* as follows. Let

$$W_\varepsilon(X, Y) := \sum_{\mathbf{x} \in S_\varepsilon} X^{\ell - \text{wt}(\mathbf{x})} Y^{\text{wt}(\mathbf{x})}, \quad (4-1)$$

where $\text{wt}(\mathbf{x})$ is the number of nonzero coordinates of \mathbf{x} . That is,

$$\text{wt}(\mathbf{x}) := \#\{x_i \mid x_i \neq 0 \text{ for } 1 \leq i \leq \ell\}.$$

We also consider a weight enumerator induced from an \mathbb{F}_4 -code \mathcal{C} . Let $W_{\mathcal{C}}(X, Y)$ denote the Hamming weight enumerator, we define

$$W'_{\mathcal{C}}(X, Y) := \frac{1}{3}(W_{\mathcal{C}}(X, Y) - X^{\ell}). \quad (4-2)$$

Note that $W_{\varepsilon}(X, Y), W'_{\mathcal{C}}(X, Y)$ are homogeneous polynomials in X, Y of the same degree ℓ .

Lemma 4.1. *The self-orthogonal \mathbb{F}_4 -code \mathcal{C} is self-dual if and only if $W'_{\mathcal{C}}(1, 1) = \frac{2^{\ell}-1}{3}$.*

Proof. First we note that $W_{\mathcal{C}}(1, 1) = |\mathcal{C}|$ is equal to the number of elements in \mathcal{C} ; hence

$$W'_{\mathcal{C}}(1, 1) = \frac{W_{\mathcal{C}}(1, 1) - 1}{3} = \frac{|\mathcal{C}| - 1}{3}.$$

Since \mathcal{C} is self-orthogonal, we know $\mathcal{C}^{\perp} \supset \mathcal{C}$ and $\dim \mathcal{C}^{\perp} + \dim \mathcal{C} = \ell$. Therefore, $|\mathcal{C}| \leq 2^{\ell}$ and the equality holds if and only if \mathcal{C} is self-dual. The lemma now follows. \square

The following lemmas explain why we introduce these weight enumerators. Recall that the module $V_{L_{\mathcal{C} \times \delta}}[\varepsilon]$ admits a decomposition of $(V_L^{\tau})^{\otimes \ell}$ -modules as

$$V_{L_{\mathcal{C} \times \delta}}[\varepsilon] \cong V_{L_{\mathbf{0} \times \delta}}[\varepsilon] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} V_{L_{\gamma \times \delta}}, \quad (4-3)$$

where $\delta \in \mathbb{Z}_3^{\ell}$ and $\mathcal{C}_{\equiv \tau}$ denotes the set of all orbits of τ in \mathcal{C} . In particular, when $\delta = \mathbf{0}$, we have $V_{L_{\mathbf{0} \times \mathbf{0}}}[0] \cong (V_L^{\tau})^{\tau}$, which should not be confused with the subVOA $(V_L^{\tau})^{\otimes \ell} \subsetneq (V_L^{\otimes \ell})^{\tau}$.

Lemma 4.2. *For $\varepsilon = 0, 1, 2$, the character of $V_{L_{\mathbf{0} \times \mathbf{0}}}[\varepsilon]$ is given by*

$$\text{ch } V_{L^{\otimes \ell}}[\varepsilon] = W_{\varepsilon}(Z_0(q), Z_1(q)),$$

where

$$Z_0(q) := \text{ch } V_L[0], \quad Z_1(q) := \text{ch } V_L[1] = \text{ch } V_L[2]. \quad (4-4)$$

Proof. For $\varepsilon = 0, 1, 2$, we have a decomposition of $(V_L^{\tau})^{\otimes \ell}$ -modules:

$$V_{L^{\otimes \ell}}[\varepsilon] = \bigoplus_{\sum r_i \equiv \varepsilon \pmod 3} V_L[r_1] \otimes \cdots \otimes V_L[r_{\ell}]. \quad (4-5)$$

We also know

$$\text{ch } (V_L[r_1] \otimes \cdots \otimes V_L[r_{\ell}]) = \text{ch } V_L[r_1] \times \cdots \times \text{ch } V_L[r_{\ell}] = Z_0^{\ell-r} Z_1^r,$$

where r is the weight of $\mathbf{r} := (r_1, \dots, r_\ell) \in \mathbb{Z}_3^\ell$. Using the definition of weight enumerator given in (4-1), we can rewrite

$$\begin{aligned} \text{ch } V_{L^{\otimes \ell}}[\varepsilon] &= \sum_{\sum r_i \equiv \varepsilon \pmod{3}} \text{ch } V_L[r_1] \times \dots \times \text{ch } V_L[r_\ell] \\ &= \sum_{\mathbf{r} \in S_\varepsilon} Z_0^{\ell - \text{wt}(\mathbf{r})} Z_1^{\text{wt}(\mathbf{r})} = W_\varepsilon(Z_0, Z_1) \end{aligned}$$

as desired. \square

Lemma 4.3. *We have the character*

$$\text{ch} \left(\bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} V_{L_{\gamma \times \mathbf{0}}} \right) = W'_{\mathcal{C}}(Y_0, Y_1),$$

where $Y_0(q) := \text{ch } V_{L(0,0)}$, and $Y_1(q) := \text{ch } V_{L(1,0)}$.

Proof. We first note that $Y_1(q) = \text{ch } V_{L(x,0)}$ for $x = 1, \omega, \bar{\omega} \in \mathbb{F}_4$. Let $\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}$. Then

$$\text{ch } V_{L_{\gamma \times \mathbf{0}}} = \prod_i \text{ch } V_{L(\gamma_i, 0)} = Y_0^{\ell - \text{wt}(\gamma)} Y_1^{\text{wt}(\gamma)}.$$

We know the τ -orbit of γ is the set $\{\gamma, \omega\gamma, \omega^2\gamma\}$, where $\omega\gamma := (\omega\gamma_1, \dots, \omega\gamma_\ell)$. Note that $\omega\gamma_i = 0$ if and only if $\gamma_i = 0$. This means $\text{wt } \gamma = \text{wt } \tau\gamma$ and hence

$$\text{ch } V_{L_{\gamma \times \mathbf{0}}} = \text{ch } V_{L_{\omega\gamma \times \mathbf{0}}} = \text{ch } V_{L_{\omega^2\gamma \times \mathbf{0}}}.$$

Therefore, by the definition of (4-2) we have

$$\text{ch} \left(\bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} V_{L_{\gamma \times \mathbf{0}}} \right) = \frac{1}{3} \sum_{\mathbf{0} \neq \gamma \in \mathcal{C}} \text{ch } V_{L_{\gamma \times \mathbf{0}}} = \frac{1}{3} \sum_{\mathbf{0} \neq \gamma \in \mathcal{C}} Y_0^{\ell - \text{wt}(\gamma)} Y_1^{\text{wt}(\gamma)} = W'_{\mathcal{C}}(Y_0, Y_1). \quad (4-6)$$

\square

Proposition 4.4. *We have*

$$\text{ch } V_{L_{\mathbf{C} \times \mathbf{0}}}[\varepsilon] = W_\varepsilon(Z_0, Z_1) + W'_{\mathcal{C}}(Y_0, Y_1),$$

for $\varepsilon = 0, 1, 2$. Moreover, we have

$$W_\varepsilon(1, 1) = 3^{\ell-1}. \quad (4-7)$$

Proof. The first statement on characters follows directly from the above two lemmas. Using a basic combinatorial argument, it is easy to show $W_i(1, 1) = 3^{\ell-1}$ for all $0 \leq i \leq 2$; note that $S_\varepsilon = (\varepsilon, \dots, 0) + S_0$ for any $\varepsilon = 1, 2$. \square

Quantum dimensions of $V_{L\mathbf{c}\times\mathbf{0}}^\tau$ -Modules. We first compute the quantum dimensions of irreducible $V_{L\mathbf{c}\times\mathbf{0}}^\tau$ -modules in the case that the code $\mathbf{D} = \{\mathbf{0}\} \in \mathbb{Z}_3^\ell$ is the trivial code. Note that in this case $\mathbf{D}^\perp = \mathbb{Z}_3^\ell$.

Proposition 4.5. *For $\varepsilon \in \mathbb{Z}_3$ and $\delta \in \mathbb{Z}_3^\ell$, the irreducible $V_{L\mathbf{c}\times\mathbf{0}}[0]$ -module $V_{L\mathbf{c}\times\delta}[\varepsilon]$ has the quantum dimension one.*

Proof. Using the same proof as in [Che13, Lemma 3.2], the irreducible $V_{L\mathbf{c}\times\mathbf{0}}^\tau$ -modules $V_{L\mathbf{c}\times\mathbf{0}}[\varepsilon]$, $\varepsilon = 0, 1, 2$, are simple current and hence have quantum dimension 1. Thus

$$1 = \text{qdim}_{V_{L\mathbf{c}\times\mathbf{0}}[0]} V_{L\mathbf{c}\times\mathbf{0}}[\varepsilon] = \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L\mathbf{c}\times\mathbf{0}}[\varepsilon]}{\text{ch } V_{L\mathbf{c}\times\mathbf{0}}[0]}.$$

If $\delta \neq \mathbf{0}$, the computation is similar with some modification.

Fix $0 \leq \varepsilon \leq 2$ and $\mathbf{0} \neq \delta \in \mathbf{D}^\perp$. Let

$$\begin{aligned} Z(q) &:= \frac{\text{ch } V_{L\mathbf{c}\times\delta}[\varepsilon]}{\text{ch } V_{L\mathbf{c}\times\mathbf{0}}[0]} = \frac{\text{ch } V_{L\mathbf{0}\times\delta}[\varepsilon] + \sum_{\mathbf{0} \neq \gamma \in \mathbf{C}_{\equiv \tau}} \text{ch } V_{L\gamma \times \delta}}{\text{ch } V_{L\mathbf{0}\times\mathbf{0}}[0] + \sum_{\mathbf{0} \neq \gamma \in \mathbf{C}_{\equiv \tau}} \text{ch } V_{L\gamma \times \mathbf{0}}} \\ &= \frac{\sum_{\mathbf{r} \in S_\varepsilon} \text{ch } V_{L(0,\delta_1)}[r_1] \times \cdots \times \text{ch } V_{L(0,\delta_\ell)}[r_\ell] + \frac{1}{3} \sum_{\mathbf{0} \neq \gamma \in \mathbf{C}} \text{ch } V_{L(r_1,\delta_1)} \times \cdots \times \text{ch } V_{L(r_\ell,\delta_\ell)}}{\sum_{\mathbf{r} \in S_0} \text{ch } V_L[r_1] \times \cdots \times \text{ch } V_L[r_\ell] + \frac{1}{3} \sum_{\mathbf{0} \neq \gamma \in \mathbf{C}} \text{ch } V_{L(r_1,0)} \times \cdots \times \text{ch } V_{L(r_\ell,0)}}. \end{aligned}$$

Dividing both denominator and numerator by $(\text{ch } V_L[0])^\ell$. We have

$$Z(q) = \frac{\sum_{\mathbf{r} \in S_\varepsilon} \frac{\text{ch } V_{L(0,\delta_1)}[r_1]}{\text{ch } V_L[0]} \times \cdots \times \frac{\text{ch } V_{L(0,\delta_\ell)}[r_\ell]}{\text{ch } V_L[0]} + \frac{1}{3} \sum_{\mathbf{0} \neq \gamma \in \mathbf{C}} \frac{\text{ch } V_{L(r_1,\delta_1)}}{\text{ch } V_L[0]} \times \cdots \times \frac{\text{ch } V_{L(r_\ell,\delta_\ell)}}{\text{ch } V_L[0]}}{\sum_{\mathbf{r} \in S_0} \frac{\text{ch } V_L[r_1]}{\text{ch } V_L[0]} \times \cdots \times \frac{\text{ch } V_L[r_\ell]}{\text{ch } V_L[0]} + \frac{1}{3} \sum_{\mathbf{0} \neq \gamma \in \mathbf{C}} \frac{\text{ch } V_{L(r_1,0)}}{\text{ch } V_L[0]} \times \cdots \times \frac{\text{ch } V_{L(r_\ell,0)}}{\text{ch } V_L[0]}}.$$

Recalling the quantum dimensions of $V_L[0]$ -modules given in Prop. 2.5, we have

$$\text{qdim}_{V_{L\mathbf{c}\times\mathbf{0}}[0]} V_{L\mathbf{c}\times\delta}[\varepsilon] = \lim_{y \rightarrow 0^+} Z(q) = \frac{W_\varepsilon(1,1) + W'_\mathbf{c}(1,1)}{W_0(1,1) + W'_\mathbf{c}(1,1)} = \frac{3^{\ell-1} + W'_\mathbf{c}(1,1)}{3^{\ell-1} + W'_\mathbf{c}(1,1)} = 1$$

as desired. \square

Proposition 4.6. *Let $\varepsilon = 0, 1, 2$ and $\boldsymbol{\eta} \in \mathbb{Z}_3^\ell$. The irreducible $V_{L\mathbf{c}\times\mathbf{0}}[0]$ -module $V_{L\mathbf{c}\times\mathbf{0}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon]$ has the quantum dimension $\frac{2^\ell}{|\mathbf{C}|}$.*

Proof. We know an irreducible $V_{L\mathbf{c}\times\mathbf{0}}[0]$ -module of twisted type admits a decomposition of $(V_L^\tau)^{\otimes \ell}$ -modules:

$$V_{L\mathbf{c}\times\mathbf{0}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] \cong \bigoplus_{\mathbf{e} \in S_\varepsilon} V_L^{T,\eta^1}(\tau^i)[e_1] \otimes \cdots \otimes V_L^{T,\eta^\ell}(\tau^i)[e_\ell]. \quad (4-8)$$

It was shown in [Che13] that $\text{ch } V_L^{T,j}(\tau^i)[1] = \text{ch } V_L^{T,j}(\tau^k)[2]$ and $\text{ch } V_L^{T,j}(\tau^i)[0] = \text{ch } V_L^{T,0}(\tau)[0]$ for $i, k = 1, 2$ and $j = 0, 1, 2$. Denote $T_0(q) := \text{ch } V_L^{T,j}(\tau^i)[0]$ and $T_1(q) := \text{ch } V_L^{T,j}(\tau^i)[1]$.

Similar to the untwisted case, we have

$$\text{ch } V_{L\mathbf{c}\times\mathbf{0}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] = W_\varepsilon(T_0, T_1). \quad (4-9)$$

Since W_ε are homogeneous polynomials of degree ℓ , we have

$$\begin{aligned} \text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau} V_{L_{\mathbf{C} \times \mathbf{0}}}^{T, \eta}(\tau^i)[\varepsilon] &= \lim_{y \rightarrow 0^+} \frac{W_\varepsilon(T_0, T_1)}{W_0(Z_0, Z_1) + W'_\mathcal{C}(Y_0, Y_1)} \\ &= \frac{W_\varepsilon(2, 2)}{W_0(1, 1) + W'_\mathcal{C}(3, 3)} = \frac{2^\ell W_\varepsilon(1, 1)}{W_0(1, 1) + 3^\ell W'_\mathcal{C}(1, 1)}. \end{aligned} \quad (4-10)$$

Note that all V_L^τ -modules $V_L^{T, j}(\tau^i)[\varepsilon]$ have quantum dimension 2.

Now by Thm. 4.4 and Lemma 4.1 we know

$$\text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau} V_{L_{\mathbf{C} \times \mathbf{0}}}^{T, \eta}(\tau^i)[\varepsilon] = \frac{2^\ell \cdot 3^{\ell-1}}{3^{\ell-1} + 3^{\ell-1}(|\mathcal{C}| - 1)} = \frac{2^\ell}{|\mathcal{C}|}.$$

□

Remark 4.7. Note that $\frac{2^\ell}{|\mathcal{C}|} = \sqrt{|\mathcal{C}^\perp / \mathcal{C}|}$ since $|\mathcal{C}^\perp| \cdot |\mathcal{C}| = \mathbb{F}_4^\ell = (2^\ell)^2$.

Corollary 4.8. *Let \mathcal{C} be a self-dual code. Then all irreducible $V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau$ -modules are simple current modules.*

Proof. If \mathcal{C} is self-dual, then $V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau$ has only two types of irreducible modules. Moreover,

$$\text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau} V_{L_{\mathbf{C} \times \mathbf{0}}}^{T, \eta}(\tau^i)[\varepsilon] = \frac{2^\ell}{|\mathcal{C}|} = 1$$

by Lemma 4.1. That means all irreducible modules of the type $V_{L_{\mathbf{C} \times \mathbf{0}}}^{T, \eta}(\tau^i)[\varepsilon]$ are simple current modules. By Proposition 4.5, the irreducible modules of the type $V_{L_{\mathbf{C} \times \delta}}[\varepsilon]$ are simple current modules, also. □

Now suppose \mathcal{C} is self-orthogonal but not self-dual. Then the quantum dimension of the $V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau$ -module $V_{L_{\mathbf{C} \times \mathbf{0}}}^{T, \eta}(\tau^i)[\varepsilon]$ is strictly greater than 1. In addition, $V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau$ has irreducible modules of the type $V_{L_{(\lambda + \mathcal{C}) \times \delta}}$.

Proposition 4.9. *Let $\lambda + \mathcal{C} \in \mathcal{C}^\perp / \mathcal{C}$ and $\delta \in \mathbb{Z}_3^\ell$, we have*

$$\text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau} V_{L_{(\lambda + \mathcal{C}) \times \delta}} = 3.$$

Proof. By definition,

$$\text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau} V_{L_{(\lambda + \mathcal{C}) \times \delta}} = \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{(\lambda + \mathcal{C}) \times \delta}}}{\text{ch } V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau} = \lim_{y \rightarrow 0^+} \frac{\sum_{\mu \in \mathcal{C}} \text{ch } V_{L_{(\lambda + \mu) \times \delta}}}{\text{ch } V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau}.$$

Dividing both the denominator and the numerator by $(\text{ch } V_L[0])^\ell$. Since $\text{qdim}_{V_L[0]} V_{L(i, j)} = 3$ for any $i \in \mathbb{F}_4, j \in \mathbb{Z}_3$, we have

$$\text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau} V_{L_{(\lambda + \mathcal{C}) \times \delta}} = \frac{|\mathcal{C}| \cdot 3^\ell}{3^{\ell-1} + 3^{\ell-1}(|\mathcal{C}| - 1)} = 3$$

as desired. □

To summarize, we have the theorem.

Theorem 4.10. *The quantum dimensions for irreducible $V_{L_{\mathbf{C} \times \mathbf{0}}}$ -modules are as follows.*

- (i) $\text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}} V_{L_{\mathbf{C} \times \delta}}[\varepsilon] = 1;$
- (ii) $\text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}} V_{L_{(\lambda + \mathbf{C}) \times \delta}} = 3;$
- (iii) $\text{qdim}_{V_{L_{\mathbf{C} \times \mathbf{0}}}} V_{L_{\mathbf{C} \times \mathbf{0}}}^{T, \eta}(\tau^i)[\varepsilon] = \frac{2^\ell}{|\mathcal{C}|},$

where $i = 1, 2$, $\varepsilon \in \mathbb{Z}_3$, $\mathbf{0} \neq \lambda + \mathbf{C} \in \mathcal{C}_{\equiv \tau}^\perp \bmod \mathcal{C}$ and $\eta, \delta \in \mathbb{Z}_3^\ell$.

Quantum dimension of $V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau$ -modules. We now deal with the general case. Let \mathcal{D} be a self-orthogonal \mathbb{Z}_3 -code. The basic idea is to express the characters of $V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau$ -modules in terms of the characters of $V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau$ -modules.

Theorem 4.11. *The quantum dimensions of irreducible $V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau$ -modules are as follows.*

- (i) $\text{qdim}_{V_{L_{\mathbf{C} \times \mathcal{D}}}} V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon] = 1;$
- (ii) $\text{qdim}_{V_{L_{\mathbf{C} \times \mathcal{D}}}} V_{L_{(\lambda + \mathbf{C}) \times (\delta + \mathcal{D})}} = 3;$
- (iii) $\text{qdim}_{V_{L_{\mathbf{C} \times \mathcal{D}}}} V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta}(\tau^i)[\varepsilon] = \frac{2^\ell}{|\mathcal{C}|},$

where $i = 1, 2$, $\varepsilon \in \mathbb{Z}_3$, $\mathbf{0} \neq \lambda + \mathbf{C} \in \mathcal{C}_{\equiv \tau}^\perp \bmod \mathcal{C}$, $\eta \in \mathcal{D}^\perp \bmod \mathcal{D}$ and $\delta + \mathcal{D} \in \mathcal{D}^\perp / \mathcal{D}$.

Proof. (i) For the module $V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon]$ we have a decomposition of $(V_L^\tau)^{\otimes \ell}$ -modules:

$$V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon] \cong V_{L_{\mathbf{0} \times (\delta + \mathcal{D})}}[\varepsilon] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} V_{L_{\gamma \times (\delta + \mathcal{D})}}. \quad (4-11)$$

Although the characters $\text{ch } V_{L_{\gamma \times \Delta}}$ may vary as Δ varies in \mathcal{D} , we still have

$$\lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\gamma \times (\delta + \Delta)}}}{\text{ch } (V_L^\tau)^{\otimes \ell}} = \prod_{i=1}^{\ell} \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L^{(\gamma^i, \delta^i + \Delta^i)}}}{\text{ch } V_L^\tau} = \prod_{i=1}^{\ell} \text{qdim } V_{L^{(\gamma^i, \delta^i + \Delta^i)}} = \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\gamma \times \delta}}}{\text{ch } (V_L^\tau)^{\otimes \ell}},$$

for all $\Delta \in \mathcal{D}$. This implies

$$\lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\gamma \times (\delta + \mathcal{D})}}}{\text{ch } (V_L^\tau)^{\otimes \ell}} = \lim_{y \rightarrow 0^+} \frac{\sum_{\Delta \in \mathcal{D}} \text{ch } V_{L_{\gamma \times \Delta}}}{\text{ch } (V_L^\tau)^{\otimes \ell}} = |\mathcal{D}| \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\gamma \times \delta}}}{\text{ch } (V_L^\tau)^{\otimes \ell}}. \quad (4-12)$$

Similarly, we have

$$\lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\mathbf{0} \times (\delta + \Delta)}}[\varepsilon]}{\text{ch } (V_L^\tau)^{\otimes \ell}} = \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\mathbf{0} \times \delta}}[\varepsilon]}{\text{ch } (V_L^\tau)^{\otimes \ell}}, \text{ for all } \Delta \in \mathcal{D}.$$

Therefore,

$$\lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\mathbf{0} \times (\delta + \mathcal{D})}}[\varepsilon]}{\text{ch } (V_L^\tau)^{\otimes \ell}} = \lim_{y \rightarrow 0^+} \frac{\sum_{\Delta \in \mathcal{D}} \text{ch } V_{L_{\mathbf{0} \times (\delta + \Delta)}}[\varepsilon]}{\text{ch } (V_L^\tau)^{\otimes \ell}} = |\mathcal{D}| \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\mathbf{0} \times \delta}}[\varepsilon]}{\text{ch } (V_L^\tau)^{\otimes \ell}}. \quad (4-13)$$

Thus by (4-11), (4-12) and (4-13) we know

$$\begin{aligned} \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon]}{\text{ch } (V_L^\tau)^{\otimes \ell}} &= \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\mathbf{0} \times (\delta + \mathcal{D})}}[\varepsilon] + \text{ch } \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} V_{L_{\gamma \times (\delta + \mathcal{D})}}}{\text{ch } (V_L^\tau)^{\otimes \ell}} \\ &= \lim_{y \rightarrow 0^+} \frac{|\mathcal{D}| \text{ch } V_{L_{\mathbf{0} \times \delta}}[\varepsilon] + |\mathcal{D}| \text{ch } \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} V_{L_{\gamma \times \delta}}}{\text{ch } (V_L^\tau)^{\otimes \ell}} = \lim_{y \rightarrow 0^+} \frac{|\mathcal{D}| \text{ch } V_{L_{\mathbf{C} \times \delta}}[\varepsilon]}{\text{ch } (V_L^\tau)^{\otimes \ell}}. \end{aligned}$$

Moreover,

$$\begin{aligned} \text{qdim}_{V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau} V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon] &= \lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon]}{\text{ch } V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau} = \lim_{y \rightarrow 0^+} \frac{\frac{1}{\text{ch}(V_L^\tau)^{\otimes \ell}} \text{ch } V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon]}{\frac{1}{\text{ch}(V_L^\tau)^{\otimes \ell}} \text{ch } V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau} \\ &= \lim_{y \rightarrow 0^+} \frac{|\mathcal{D}| \text{ch } V_{L_{\mathbf{C} \times \delta}}[\varepsilon]}{|\mathcal{D}| \text{ch } V_{L_{\mathbf{C} \times 0}}[0]} = \text{qdim}_{V_{L_{\mathbf{C} \times 0}}^\tau} V_{L_{\mathbf{C} \times \delta}}[\varepsilon] = 1. \end{aligned}$$

(ii) By the similar arguments as (i), we have

$$\lim_{y \rightarrow 0^+} \frac{\text{ch } V_{L_{(\lambda + \mathbf{C}) \times (\delta + \mathcal{D})}}}{\text{ch}(V_L^\tau)^{\otimes \ell}} = \lim_{y \rightarrow 0^+} \frac{\text{ch} \left(\bigoplus_{\Delta \in \mathcal{D}} V_{L_{(\lambda + \mathbf{C}) \times (\delta + \Delta)}} \right)}{\text{ch}(V_L^\tau)^{\otimes \ell}} = \lim_{y \rightarrow 0^+} \frac{|\mathcal{D}| \text{ch } V_{L_{(\lambda + \mathbf{C}) \times \delta}}}{\text{ch}(V_L^\tau)^{\otimes \ell}},$$

and hence

$$\text{qdim}_{V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau} V_{L_{(\lambda + \mathbf{C}) \times (\delta + \mathcal{D})}} = \text{qdim}_{V_{L_{\mathbf{C} \times 0}}^\tau} V_{L_{(\lambda + \mathbf{C}) \times \delta}} = 3.$$

(iii) For the irreducible $V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau$ -modules of the type $V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta}(\tau^i)[\varepsilon]$, we have the decomposition of $(V_L^\tau)^{\otimes \ell}$ -modules:

$$V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta}(\tau^i)[\varepsilon] \cong \bigoplus_{\gamma \in \mathcal{D}} \bigoplus_{\mathbf{e} \in S_\varepsilon} V_L^{T, \eta_1 - i\gamma_1}(\tau^i)[e_1] \otimes \cdots \otimes V_L^{T, \eta_\ell - i\gamma_\ell}(\tau^i)[e_\ell].$$

Fix $\mathbf{e} \in \mathbb{Z}_3^\ell$; the characters $\text{ch } V_L^{T, \eta_1 - i\gamma_1}(\tau^i)[e_1] \otimes \cdots \otimes V_L^{T, \eta_\ell - i\gamma_\ell}(\tau^i)[e_\ell]$ are all the same for any $(\gamma_1, \dots, \gamma_\ell) \in \mathcal{D}$. Thus,

$$\begin{aligned} \text{ch } V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta}(\tau^i)[\varepsilon] &= |\mathcal{D}| \bigoplus_{\mathbf{e} \in S_\varepsilon} \text{ch } V_L^{T, \eta_1 - i\gamma_1}(\tau^i)[e_1] \otimes \cdots \otimes V_L^{T, \eta_\ell - i\gamma_\ell}(\tau^i)[e_\ell] \\ &= |\mathcal{D}| \text{ch } V_{L_{\mathbf{C} \times 0}}^{T, \eta}(\tau^i)[\varepsilon]. \end{aligned}$$

As before we have

$$\text{qdim}_{V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau} V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta}(\tau^i)[\varepsilon] = \text{qdim}_{V_{L_{\mathbf{C} \times 0}}^\tau} V_{L_{\mathbf{C} \times 0}}^{T, \eta}(\tau^i)[\varepsilon],$$

which is $2^\ell / |\mathcal{C}|$. □

Global Dimension. Let V be a VOA with only finitely many irreducible modules, the *global dimension* of V [DJX13] is defined as

$$\text{glob}(V) := \sum_{M \in \text{Irr}(V)} \text{qdim}(M)^2. \quad (4-14)$$

Assume G is a finite subgroup of $\text{Aut}(V)$, it is conjectured in [DJX13] that

$$|G|^2 \text{glob}(V) = \text{glob}(V^G).$$

We will verify this conjecture in our case, *i.e.*, $V = V_{L_{\mathbf{C} \times \mathcal{D}}}$ and $G = \langle \tau \rangle$.

Since all irreducible $V_{L_{\mathbf{C} \times \mathcal{D}}}$ -modules are simple current, we have

$$\text{glob}(V_{L_{\mathbf{C} \times \mathcal{D}}}) = |\mathcal{C}^\perp / \mathcal{C}| |\mathcal{D}^\perp / \mathcal{D}| \cdot 1^2.$$

The global dimension of $V_{L_{\mathcal{C} \times \mathcal{D}}}^T$ will be computed below. We count the number of irreducibles that have the same quantum dimensions.

- (i) $\text{qdim}_{V_{L_{\mathcal{C} \times \mathcal{D}}}^T} V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon] = 1$. There are $|\mathcal{D}^\perp / \mathcal{D}| \cdot 3$ irreducible modules of this type.
- (ii) $\text{qdim}_{V_{L_{\mathcal{C} \times \mathcal{D}}}^T} V_{L_{(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})}} = 3$ if $\mathbf{0} \neq \lambda + \mathcal{C} \in \mathcal{C}_{\equiv \tau}^\perp \bmod \mathcal{C}$. There are $|\mathcal{D}^\perp / \mathcal{D}| \cdot \frac{|\mathcal{C}^\perp / \mathcal{C}| - 1}{3}$ irreducible modules of this type.
- (iii) $\text{qdim}_{V_{L_{\mathcal{C} \times \mathcal{D}}}^T} V_{L_{\mathcal{C} \times \mathbf{0}}}^{T, \eta}(\tau^i)[\varepsilon] = \frac{2^\ell}{|\mathcal{C}|}$. There are $|\mathcal{D}^\perp / \mathcal{D}| \cdot 3 \cdot 2$ irreducible modules of this type.

Note that $(2^\ell / |\mathcal{C}|)^2 = |\mathcal{C}^\perp / \mathcal{C}|$. Therefore,

$$\text{glob } V_{L_{\mathcal{C} \times \mathcal{D}}}^T = |\mathcal{D}^\perp / \mathcal{D}| \left(3 + \frac{|\mathcal{C}^\perp / \mathcal{C}| - 1}{3} \cdot 3^2 + 6 |\mathcal{C}^\perp / \mathcal{C}| \right) = 9 |\mathcal{C}^\perp / \mathcal{C}| |\mathcal{D}^\perp / \mathcal{D}|.$$

Hence we have $\text{glob}(V_{L_{\mathcal{C} \times \mathcal{D}}}) \cdot 3^2 = \text{glob}(V_{L_{\mathcal{C} \times \mathcal{D}}}^T)$. This verified the conjecture of Dong, Jiao and Xu in this special case.

5. FUSION RULES

In this section, we compute the fusion rules of $V_{L_{\mathcal{C} \times \mathcal{D}}}^T$ -modules. The next three propositions are crucial to our calculations.

Proposition 5.1 ([TY13, Prop.4.5]). *Let $\varepsilon, \varepsilon_1, \varepsilon_2, j, j_1, j_2, k \in \mathbb{Z}_3$ and $i = 1, 2$. Then*

- (i) $V_{L(0, j_1)}[\varepsilon_1] \times V_{L(0, j_2)}[\varepsilon_2] = V_{L(0, j_1 + j_2)}[\varepsilon_1 + \varepsilon_2];$
- (ii) $V_{L(0, j_1)}[\varepsilon] \times V_{L(c, j_2)} = V_{L(c, j_1 + j_2)};$
- (iii) $V_{L(c, j_1)} \times V_{L(c, j_2)} = \sum_{\rho=0}^2 V_{L(0, j_1 + j_2)}[\rho] + 2V_{L(c, j_1 + j_2)};$
- (iv) $V_{L(0, j)}[\varepsilon_1] \times V_L^{T, k}(\tau^i)[\varepsilon_2] = V_L^{T, k-ij}(\tau^i)[i\varepsilon_1 + \varepsilon_2];$
- (v) $V_{L(c, j)} \times V_L^{T, k}(\tau^i)[\varepsilon] = \sum_{\rho=0}^2 V_L^{T, k-ij}(\tau^i)[\rho].$

Proposition 5.2. [Che13] *We have the following fusion rules among irreducible V_L^T -modules of twisted type.*

- (i) $V_L^{T, i}(\tau^l)[\varepsilon] \times V_L^{T, j}(\tau^l)[\varepsilon'] = V_L^{T, -(i+j)}(\tau^{2l})[-(\varepsilon + \varepsilon')] + V_L^{T, -(i+j)}(\tau^{2l})[2 - (\varepsilon + \varepsilon')];$
- (ii) $V_L^{T, i}(\tau)[\varepsilon] \times V_L^{T, j}(\tau^2)[\varepsilon'] = V_{L(0, i+2j)}[\varepsilon + 2\varepsilon'] + V_{L(c, i+2j)},$

where $l \in \{1, 2\}$, $i, j, \varepsilon, \varepsilon' \in \{0, 1, 2\}$.

Proposition 5.3 ([ADL05, Prop 2.9]). *Let V be a vertex operator algebra and let M^1, M^2, M^3 be V -modules among which M^1 and M^2 are irreducible. Suppose that U is a vertex operator subalgebra of V (with the same Virasoro element) and that N^1 and N^2 are irreducible U -submodules of M^1 and M^2 , respectively. Then the restriction map from $I_V\left(\begin{smallmatrix} M^3 \\ M^1, M^2 \end{smallmatrix}\right)$ to $I_U\left(\begin{smallmatrix} M^3 \\ N^1, N^2 \end{smallmatrix}\right)$ is injective. In particular,*

$$\dim I_V\left(\begin{smallmatrix} M^3 \\ M^1, M^2 \end{smallmatrix}\right) \leq \dim I_U\left(\begin{smallmatrix} M^3 \\ N^1, N^2 \end{smallmatrix}\right). \quad (5-1)$$

In our case, we consider the following chain of subVOAs:

$$V_{L_{\mathcal{C} \times \mathcal{D}}} \supset V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau \supset V_{L_{\mathcal{C} \times 0}}^\tau \supset (V_L^\tau)^{\otimes \ell}.$$

For simplicity, we denote

$$\begin{aligned} N_{\mathcal{C} \times \mathcal{D}} \left(\begin{array}{c} - \\ - \end{array} \right) &= \dim I_{V_{L_{\mathcal{C} \times \mathcal{D}}}} \left(\begin{array}{c} - \\ - \end{array} \right), & N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{array}{c} - \\ - \end{array} \right) &= \dim I_{V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau} \left(\begin{array}{c} - \\ - \end{array} \right), \\ N_{\otimes} \left(\begin{array}{c} - \\ - \end{array} \right) &= \dim I_{(V_L^\tau)^{\otimes \ell}} \left(\begin{array}{c} - \\ - \end{array} \right), & N_{\circ}^\tau \left(\begin{array}{c} - \\ - \end{array} \right) &= \dim I_{V_L^\tau} \left(\begin{array}{c} - \\ - \end{array} \right). \end{aligned}$$

The basic idea is to use Proposition 5.3 and the quantum dimensions of V_L^τ -modules to show that many fusion coefficients are zero. This gives some inequalities on fusion rules. Next by using quantum dimensions, we show that these inequalities are actually equalities.

Let $\lambda + \mathcal{C}, \lambda^1 + \mathcal{C}, \lambda^2 + \mathcal{C} \in \mathcal{C}^\perp / \mathcal{C}$, $\delta + \mathcal{D}, \delta^1 + \mathcal{D}, \delta^2 + \mathcal{D} \in \mathcal{D}^\perp / \mathcal{D}$, $\eta, \eta^1, \eta^2 \in \mathcal{D}^\perp \bmod \mathcal{D}$ and $\varepsilon, \varepsilon^1, \varepsilon^2 \in \mathbb{Z}_3$. We will compute fusion rules separately in the following cases:

- (I) Fusion rules of the form $V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon] \times M$ for any irreducible module M (see Prop. 5.4);
- (II) Fusion rules of the form $V_{L_{(\lambda^1 + \mathcal{C}) \times (\delta^1 + \mathcal{D})}} \times V_{L_{(\lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}}$ (see Prop. 5.6);
- (III) Fusion rules of the form $V_{L_{(\lambda + \mathcal{C})}} \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta}(\tau^i)[\varepsilon]$ (see Prop. 5.8);
- (IV) Fusion rules of the form $V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^1}(\tau)[\varepsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^2}(\tau^2)[\varepsilon^2]$ (see Prop. 5.9);
- (V) Fusion rules of the form $V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^1}(\tau^i)[\varepsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^2}(\tau^i)[\varepsilon^2]$. In this case, we first determine the fusion coefficients up to a permutation in Prop. 5.10. Then we use modular invariance property of trace functions to get an explicit result in Prop. 5.17.

We start with Case (I).

Proposition 5.4. *We have the following fusion rules.*

$$(i) \quad V_{L_{\mathcal{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathcal{C} \times (\delta^2 + \mathcal{D})}}[\varepsilon^2] = V_{L_{\mathcal{C} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon^1 + \varepsilon^2]; \quad (5-2a)$$

$$(ii) \quad V_{L_{\mathcal{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{(\lambda + \mathcal{C}) \times (\delta^2 + \mathcal{D})}} = V_{L_{(\lambda + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}}; \quad (5-2b)$$

$$(iii) \quad V_{L_{\mathcal{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^2 - i\delta^1}(\tau^i)[i\varepsilon^1 + \varepsilon^2], \quad (5-2c)$$

where $\delta^1 + \mathcal{D}, \delta^2 + \mathcal{D} \in \mathcal{D}^\perp / \mathcal{D}$, $0 \neq \lambda + \mathcal{C} \in \mathcal{C}_{\equiv \tau}^\perp \bmod \mathcal{C}$ and $\varepsilon^1, \varepsilon^2 \in \mathbb{Z}_3$.

Proof. (i) Observe that $\text{qdim}(V_{L_{\mathcal{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathcal{C} \times (\delta^2 + \mathcal{D})}}[\varepsilon^2]) = 1$; therefore the fusion product $V_{L_{\mathcal{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathcal{C} \times (\delta^2 + \mathcal{D})}}[\varepsilon^2]$ is irreducible.

Recall the fusion rules of $V_{L_{\mathcal{C} \times \mathcal{D}}}$ -modules:

$$1 = N_{\mathcal{C} \times \mathcal{D}} \left(\begin{array}{c} V_{L_{\mathcal{C} \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ V_{L_{\mathcal{C} \times (\delta^1 + \mathcal{D})}}, V_{L_{\mathcal{C} \times (\delta^2 + \mathcal{D})}} \end{array} \right).$$

By restricting to $V_{L_{\mathbf{C} \times \mathcal{D}}}^\tau$ -modules and using Prop. 5.3, we have

$$1 \leq N_{\mathbf{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{\mathbf{C} \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1], V_{L_{\mathbf{C} \times (\delta^2 + \mathcal{D})}}[\varepsilon^2] \end{matrix} \right) = \sum_{\varepsilon=0}^2 N_{\mathbf{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{\mathbf{C} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon] \\ V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1], V_{L_{\mathbf{C} \times (\delta^2 + \mathcal{D})}}[\varepsilon^2] \end{matrix} \right).$$

Therefore, we know

$$V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathbf{C} \times (\delta^2 + \mathcal{D})}}[\varepsilon^2] = V_{L_{\mathbf{C} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon],$$

for some $\varepsilon \in \mathbb{Z}_3$. For simplicity, we let $M^i := V_{L_{\mathbf{C} \times (\delta^i + \mathcal{D})}}[\varepsilon^i]$ and $M := V_{L_{\mathbf{C} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon]$.

Recall the decompositions

$$V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon] = V_{L_{\mathbf{0} \times (\delta + \mathcal{D})}}[\varepsilon] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} V_{L_{\gamma \times (\delta + \mathcal{D})}};$$

$$V_{L_{\mathbf{0} \times \delta}}[\varepsilon] = \bigoplus_{\mathbf{e} \in S_\varepsilon} V_{L_{(0, \delta_1^i)}}[e_1^i] \otimes \cdots \otimes V_{L_{(0, \delta_\ell^i)}}[e_\ell^i].$$

Now fix an irreducible $(V_L^\tau)^{\otimes \ell}$ -submodule

$$N^i := V_{L_{(0, \delta_1^i)}}[e_1^i] \otimes \cdots \otimes V_{L_{(0, \delta_\ell^i)}}[e_\ell^i] \subset V_{L_{\mathbf{0} \times (\delta^i + \mathcal{D})}}[\varepsilon^i] \subset M^i$$

for some $\mathbf{e}^i := (e_1^i, \dots, e_\ell^i) \in S_{\varepsilon^i}$. Since

$$M := V_{L_{\mathbf{C} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon] \cong V_{L_{\mathbf{0} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} V_{L_{\gamma \times (\delta^1 + \delta^2 + \mathcal{D})}}, \quad (5-3)$$

we have the fusion coefficient

$$1 = N_{\mathbf{C} \times \mathcal{D}}^\tau \left(\begin{matrix} M \\ M^1, M^2 \end{matrix} \right) \leq N_{\otimes} \left(\begin{matrix} V_{L_{\mathbf{0} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon] \\ N^1, N^2 \end{matrix} \right) + \sum_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}} N_{\otimes} \left(\begin{matrix} V_{L_{\gamma \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ N^1, N^2 \end{matrix} \right). \quad (5-4)$$

We claim that

$$N_{\otimes} \left(\begin{matrix} V_{L_{\gamma \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ N^1, N^2 \end{matrix} \right) = 0 \quad \text{for all } \mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}.$$

We have

$$N_{\otimes} \left(\begin{matrix} V_{L_{\gamma \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ N^1, N^2 \end{matrix} \right) = \sum_{\Delta \in \mathcal{D}} \prod_{k=1}^{\ell} N_{\circ}^\tau \left(\begin{matrix} V_{L_{(\gamma_k, \delta_k^1 + \delta_k^2 + \Delta_k)}} \\ V_{L_{(0, \delta_k^1)}}[e_k^1], V_{L_{(0, \delta_k^2)}}[e_k^2] \end{matrix} \right). \quad (5-5)$$

Now, since $\gamma \neq \mathbf{0}$ we have $\gamma_h \neq 0$ for some $1 \leq h \leq \ell$ and hence

$$N_{\circ}^\tau \left(\begin{matrix} V_{L_{(\gamma_h, \delta_h^1 + \delta_h^2 + \Delta_h)}} \\ V_{L_{(0, \delta_h^1)}}[e_h^1], V_{L_{(0, \delta_h^2)}}[e_h^2] \end{matrix} \right) = 0.$$

This proves our claim and equation (5-4) becomes

$$1 \leq N_{\otimes} \left(\begin{matrix} V_{L_{\mathbf{0} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon] \\ N^1, N^2 \end{matrix} \right). \quad (5-6)$$

Now, we set $(e_1^i, \dots, e_\ell^i) = (\varepsilon^i, 0, \dots, 0)$ for $i = 1, 2$. Then we have

$$\begin{aligned} 1 &\leq N_{\otimes} \left(\begin{matrix} V_{L_{\mathbf{0} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon] \\ N^1, N^2 \end{matrix} \right) = \sum_{\Delta \in \mathcal{D}} N_{\otimes} \left(\begin{matrix} V_{L_{\mathbf{0} \times (\delta^1 + \delta^2 + \Delta)}}[\varepsilon] \\ N^1, N^2 \end{matrix} \right) \\ &= \sum_{\Delta \in \mathcal{D}} N_{\otimes} \left(\begin{matrix} \sum_{\mathbf{r} \in S_\varepsilon} V_{L_{(0, \delta_1^1 + \delta_1^2 + \Delta_1)}}[r_1] \otimes \dots \otimes V_{L_{(0, \delta_\ell^1 + \delta_\ell^2 + \Delta_\ell)}}[r_\ell] \\ V_{L_{(0, \delta_1^1)}}[\varepsilon^1] \otimes V_{L_{(0, \delta_2^1)}}[0] \otimes \dots \otimes V_{L_{(0, \delta_\ell^1)}}[0], N^2 \end{matrix} \right) \\ &= \sum_{\substack{\mathbf{r} \in S_\varepsilon \\ \Delta \in \mathcal{D}}} \left(N_{\circ}^{\tau} \left(\begin{matrix} V_{L_{(0, \delta_1^1 + \delta_1^2 + \Delta_1)}}[r_1] \\ V_{L_{(0, \delta_1^1)}}[\varepsilon^1], V_{L_{(0, \delta_1^2)}}[\varepsilon^2] \end{matrix} \right) \prod_{k=2}^{\ell} N_{\circ}^{\tau} \left(\begin{matrix} V_{L_{(0, \delta_k^1 + \delta_k^2 + \Delta_k)}}[r_k] \\ V_{L_{(0, \delta_k^1)}}[0], V_{L_{(0, \delta_k^2)}}[0] \end{matrix} \right) \right). \end{aligned}$$

By Prop. 5.1 we know if $(r_2, \dots, r_\ell) \neq (0, \dots, 0)$ then

$$N_{\circ}^{\tau} \left(\begin{matrix} V_{L_{(0, \delta_1^1 + \delta_1^2 + \Delta_1)}}[r_1] \\ V_{L_{(0, \delta_1^1)}}[\varepsilon^1], V_{L_{(0, \delta_1^2)}}[\varepsilon^2] \end{matrix} \right) \prod_{k=2}^{\ell} N_{\circ}^{\tau} \left(\begin{matrix} V_{L_{(0, \delta_k^1 + \delta_k^2 + \Delta_k)}}[r_k] \\ V_{L_{(0, \delta_k^1)}}[0], V_{L_{(0, \delta_k^2)}}[0] \end{matrix} \right) = 0.$$

Thus only $\mathbf{r} = (r_1, 0, \dots, 0) \in S_\varepsilon$ contributes a nonzero summand. Therefore

$$\begin{aligned} 1 &\leq N_{\otimes} \left(\begin{matrix} V_{L_{\mathbf{0} \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon] \\ N^1, N^2 \end{matrix} \right) \\ &= \sum_{\Delta \in \mathcal{D}} \left(N_{\circ}^{\tau} \left(\begin{matrix} V_{L_{(0, \delta_1^1 + \delta_1^2 + \Delta_1)}}[r_1] \\ V_{L_{(0, \delta_1^1)}}[\varepsilon^1], V_{L_{(0, \delta_1^2)}}[\varepsilon^2] \end{matrix} \right) \prod_{k=2}^{\ell} N_{\circ}^{\tau} \left(\begin{matrix} V_{L_{(0, \delta_k^1 + \delta_k^2 + \Delta_k)}}[0] \\ V_{L_{(0, \delta_k^1)}}[0], V_{L_{(0, \delta_k^2)}}[0] \end{matrix} \right) \right). \end{aligned}$$

Since $\mathbf{r} \in S_\varepsilon$, we must have $r_1 = \varepsilon = \varepsilon^1 + \varepsilon^2$. This proves (i).

(ii) We know the fusion coefficient of $V_{L_{\mathbf{C} \times \mathcal{D}}}$ -modules:

$$1 = N_{\mathbf{C} \times \mathcal{D}} \left(\begin{matrix} V_{L_{(\lambda + \mathbf{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}, V_{L_{(\lambda + \mathbf{C}) \times (\delta^2 + \mathcal{D})}} \end{matrix} \right).$$

By restricting to $V_{L_{\mathbf{C} \times \mathcal{D}}}^{\tau}$ -modules, we have

$$1 \leq N_{\mathbf{C} \times \mathcal{D}}^{\tau} \left(\begin{matrix} V_{L_{(\lambda + \mathbf{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1], V_{L_{(\lambda + \mathbf{C}) \times (\delta^2 + \mathcal{D})}} \end{matrix} \right).$$

Since $\text{qdim } V_{L_{(\lambda + \mathbf{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}} = \text{qdim } (V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{(\lambda + \mathbf{C}) \times (\delta^2 + \mathcal{D})}})$, we prove (ii).

(iii) Since $V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1]$ is simple current and $\text{qdim } (V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2]) = \frac{2^\ell}{|\mathcal{C}|}$, we know the fusion product $V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2]$ is either $V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta}(\tau^j)[\varepsilon]$ for some $\delta + \mathbf{C} \in \mathcal{C}^\perp / \mathcal{C}, \varepsilon \in \mathbb{Z}_3$ and $j = 1, 2$ or $V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon]$ if $|\mathcal{C}| = 2^\ell$.

Assume

$$V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon].$$

Then we have

$$V_{L_{\mathbf{C} \times (-\delta^1 + \mathcal{D})}}[-\varepsilon^1] \times V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L_{\mathbf{C} \times (-\delta^1 + \mathcal{D})}}[-\varepsilon^1] \times V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon],$$

and hence by (5-2a)

$$V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L_{\mathbf{C} \times \mathcal{D}}}[0] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L_{\mathbf{C} \times (\delta - \delta^1 + \mathcal{D})}}[\varepsilon - \varepsilon^1],$$

a contradiction. Therefore,

$$V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^3}(\tau^j)[\varepsilon^3],$$

for some $j = 1, 2$, $\delta^h + \mathbf{C} \in \mathbf{C}^\perp / \mathbf{C}$, $\varepsilon^h \in \mathbb{Z}_3$, for $h = 1, 2, 3$.

Similar to (i), we pick the following irreducible $(V_L^T)^{\otimes \ell}$ -modules

$$\begin{aligned} V_{L_{(0, \delta_1^1)}}[e_1^1] \otimes \cdots \otimes V_{L_{(0, \delta_\ell^1)}}[e_\ell^1] &\subset V_{L_{\mathbf{0} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1]; \\ V_{L_{(0, \delta_1^2)}}(\tau^i)[e_1^2] \otimes \cdots \otimes V_{L_{(0, \delta_\ell^2)}}(\tau^i)[e_\ell^2] &\subset V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2]; \end{aligned}$$

of M^i for some $\mathbf{e}^h := (e_1^h, \dots, e_\ell^h) \in S_{\varepsilon_h}$, $h = 1, 2$.

Prop. 5.3 suggests that

$$\begin{aligned} 1 &= N_{\mathbf{C} \times \mathcal{D}}^T \left(V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^3}(\tau^j)[\varepsilon^3], V_{L_{\mathbf{C} \times (\delta^1 + \mathcal{D})}}[\varepsilon^1], V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2] \right) \\ &\leq N^\circ \left(\sum_{\mathbf{e}^3 \in S_{\varepsilon_3}} V_{L_{(0, \delta_1^3)}}^{T, \delta_1^3}(\tau^i)[e_1^3] \otimes \cdots \otimes V_{L_{(0, \delta_\ell^3)}}^{T, \delta_\ell^3}(\tau^i)[e_\ell^3], V_{L_{(0, \delta_1^1)}}[e_1^1] \otimes \cdots \otimes V_{L_{(0, \delta_\ell^1)}}[e_\ell^1], V_{L_{(0, \delta_1^2)}}(\tau^i)[e_1^2] \otimes \cdots \otimes V_{L_{(0, \delta_\ell^2)}}(\tau^i)[e_\ell^2] \right) \\ &= \sum_{\mathbf{e}^3 \in S_{\varepsilon_3}} \prod_{k=1}^{\ell} N_{\circ}^T \left(V_{L_{(0, \delta_k^3)}}^{T, \delta_k^3}(\tau^j)[e_k^3], V_{L_{(0, \delta_k^1)}}[e_k^1], V_{L_{(0, \delta_k^2)}}^{T, \delta_k^2}(\tau^i)[e_k^2] \right). \end{aligned}$$

If $j \neq i$, then Prop. 5.1 gives $1 \leq 0$, a contradiction. Therefore $j = i$. If there exists $1 \leq k \leq \ell$ such that $\delta_k^3 \neq \delta_k^2 - i\delta_k^1$ or $e_k^3 \neq ie_k^1 + e_k^2$, again Prop. 5.1 gives $1 \leq 0$, a contradiction.

Therefore, we must have $\delta_k^3 = \delta_k^2 - i\delta_k^1$ and $e_k^3 = ie_k^1 + e_k^2$ for all k . This gives $\delta^3 = \delta^2 - i\delta^1$ and $\varepsilon_3 \equiv \sum_{k=1}^{\ell} e_k^3 = \sum_{k=1}^{\ell} ie_k^1 + e_k^2 \equiv i\varepsilon^1 + \varepsilon^2 \pmod{3}$. This completes the proof. \square

Using the above proposition, we can find the contragredient dual of irreducible modules. Recall there are natural isomorphisms between the following fusion rules:

$$N \left(\begin{smallmatrix} C \\ A, B \end{smallmatrix} \right) = N \left(\begin{smallmatrix} C \\ B, A \end{smallmatrix} \right) = N \left(\begin{smallmatrix} B' \\ A, C' \end{smallmatrix} \right),$$

for every V -modules A, B and C .

Proposition 5.5. *The contragredient dual of irreducible $V_{L_{\mathbf{C} \times \mathcal{D}}}^T$ -modules is listed below.*

- (i) $(V_{L_{\mathbf{C} \times (\delta + \mathcal{D})}}[\varepsilon])' = V_{L_{\mathbf{C} \times (-\delta + \mathcal{D})}}[-\varepsilon];$
- (ii) $(V_{L_{(\lambda + \mathbf{C}) \times (\delta + \mathcal{D})}})' = V_{L_{(-\lambda + \mathbf{C}) \times (-\delta + \mathcal{D})}} = V_{L_{(\lambda + \mathbf{C}) \times (-\delta + \mathcal{D})}};$
- (iii) $(V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta}(\tau^i)[\varepsilon])' = V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta}(\tau^{2i})[\varepsilon].$

Proof. It is discussed in Prop. 3.8 that $(V_{L_{\mathcal{C} \times \mathcal{D}}}[0])' \cong V_{L_{\mathcal{C} \times \mathcal{D}}}[0]$ is self-dual. We know the fusion rule:

$$1 = N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon] \\ V_{L_{\mathcal{C} \times \mathcal{D}}}[0], V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon] \end{matrix} \right) = N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} (V_{L_{\mathcal{C} \times \mathcal{D}}}[0])' \\ (V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon])', V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon] \end{matrix} \right).$$

Since $\text{qdim } M = \text{qdim } M'$ for any module M , by Prop. 5.4 we may assume $(V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon])' \cong V_{L_{\mathcal{C} \times (\delta' + \mathcal{D})}}[\varepsilon']$ for some δ', ε' . Now using equation (5-2a) we must have

$$(V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon])' = V_{L_{\mathcal{C} \times (-\delta + \mathcal{D})}}[-\varepsilon].$$

This proves (i). Similarly, using equation (5-2b) we have (ii).

(iii) We take a different approach. We first consider the contragredient dual of the irreducible V_L^τ -modules of twisted type. We know V_L^τ is self-dual. Let $i, \varepsilon \in \mathbb{Z}_3$, then

$$1 = N_{\circ}^\tau \left(\begin{matrix} V_L^{T,i}(\tau^l)[\varepsilon] \\ V_L^\tau, V_L^{T,i}(\tau^l)[\varepsilon] \end{matrix} \right) = N_{\circ}^\tau \left(\begin{matrix} V_L^\tau \\ V_L^{T,i}(\tau^l)[\varepsilon], (V_L^{T,i}(\tau^l)[\varepsilon])' \end{matrix} \right).$$

By quantum dimensions and fusion rules of V_L^τ -modules, we must have

$$(V_L^{T,i}(\tau^l)[\varepsilon])' = V_L^{T,i}(\tau^{2l})[\varepsilon].$$

Now, consider the decomposition of $(V_L^\tau)^{\otimes \ell}$ -modules:

$$V_{L_{\mathcal{C} \times \mathcal{D}}}^{T,\eta}(\tau^i)[\varepsilon] \cong \bigoplus_{\delta \in \mathcal{D}} \bigoplus_{e_1 + \dots + e_\ell \equiv \varepsilon \pmod{3}} V_L^{T,\eta_1 - i\delta_1}(\tau^i)[e_1] \otimes \dots \otimes V_L^{T,\eta_\ell - i\delta_\ell}(\tau^i)[e_\ell].$$

Taking contragredient dual as $(V_L^\tau)^{\otimes \ell}$ -modules, we have

$$(V_{L_{\mathcal{C} \times \mathcal{D}}}^{T,\eta}(\tau^i)[\varepsilon])' \cong \bigoplus_{\delta \in \mathcal{D}} \bigoplus_{e_1 + \dots + e_\ell \equiv \varepsilon \pmod{3}} V_L^{T,\eta_1 - i\delta_1}(\tau^{2i})[e_1] \otimes \dots \otimes V_L^{T,\eta_\ell - i\delta_\ell}(\tau^{2i})[e_\ell].$$

Since $V_{L_{\mathcal{C} \times \mathcal{D}}}^{T,\eta}(\tau^{2i})[\varepsilon]$ is the only irreducible $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ -module admitting the above decomposition of $(V_L^\tau)^{\otimes \ell}$ -modules, we must have

$$(V_{L_{\mathcal{C} \times \mathcal{D}}}^{T,\eta}(\tau^i)[\varepsilon])' \cong V_{L_{\mathcal{C} \times \mathcal{D}}}^{T,\eta}(\tau^{2i})[\varepsilon].$$

□

Case (II): $V_{L_{(\lambda^1 + \mathcal{C}) \times (\delta^1 + \mathcal{D})}} \times V_{L_{(\lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}}$

Proposition 5.6. *We have the fusion rules*

$$V_{L_{(\lambda^1 + \mathcal{C}) \times (\delta^1 + \mathcal{D})}} \times V_{L_{(\lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}} = \bigoplus_{h=0}^2 V_{L_{(\lambda^1 + \omega^h \lambda^2 + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}}.$$

Proof. Fix $0 \leq h \leq 2$ we have the fusion rules of $V_{L_{\mathcal{C} \times \mathcal{D}}}$ -modules:

$$1 = N_{\mathcal{C} \times \mathcal{D}} \left(\begin{matrix} V_{L_{(\lambda^1 + \omega^h \lambda^2 + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ V_{L_{(\lambda^1 + \mathcal{C}) \times (\delta^1 + \mathcal{D})}}, V_{L_{(\omega^h \lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}} \end{matrix} \right) \leq N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{(\lambda^1 + \omega^h \lambda^2 + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ V_{L_{(\lambda^1 + \mathcal{C}) \times (\delta^1 + \mathcal{D})}}, V_{L_{(\omega^h \lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}} \end{matrix} \right).$$

Since $\omega^h \lambda^2 + \mathcal{C}$, $0 \leq h \leq 2$, are identical in $\mathcal{C}_{\equiv \tau}^\perp \bmod \mathcal{C}$, there is an isomorphism of $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ -modules

$$V_{L_{(\lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}} \cong V_{L_{(\omega \lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}} \cong V_{L_{(\omega^2 \lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}}.$$

Therefore, we can write

$$1 \leq N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{(\lambda^1 + \omega^h \lambda^2 + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}} \\ V_{L_{(\lambda^1 + \mathcal{C}) \times (\delta^1 + \mathcal{D})}}, V_{L_{(\lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}} \end{matrix} \right),$$

for all h . Since $\lambda^1 + \omega^h \lambda^2 + \mathcal{C}$, $0 \leq h \leq 2$, are distinct in $\mathcal{C}_{\equiv \tau}^\perp \bmod \mathcal{C}$, by counting quantum dimensions, we prove

$$V_{L_{(\lambda^1 + \mathcal{C}) \times (\delta^1 + \mathcal{D})}} \times V_{L_{(\lambda^2 + \mathcal{C}) \times (\delta^2 + \mathcal{D})}} = \bigoplus_{h=0}^2 V_{L_{(\lambda^1 + \omega^h \lambda^2 + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}}.$$

This completes the proof. \square

Remark 5.7. Note that if $\lambda^1 + \omega^h \lambda^2 = 0$ for some h , then the module $V_{L_{(\lambda^1 + \omega^h \lambda^2 + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}}$ is not irreducible and admits a decomposition of irreducible modules of $V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau$ -modules:

$$V_{L_{(\lambda^1 + \omega^h \lambda^2 + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}} = \sum_{\varepsilon=0}^2 V_{L_{(\lambda^1 + \omega^h \lambda^2 + \mathcal{C}) \times (\delta^1 + \delta^2 + \mathcal{D})}}[\varepsilon].$$

Case (III): $V_{L_{(\gamma + \mathcal{C}) \times (\delta^1 + \mathcal{D})}} \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon]$

Proposition 5.8. *We have*

$$(i) \quad V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}} \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^i)[0] = \bigoplus_{\rho=0}^2 V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^i)[\rho]; \quad (5-7a)$$

$$(ii) \quad V_{L_{(\gamma + \mathcal{C}) \times (\delta^1 + \mathcal{D})}} \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon] = \bigoplus_{\rho=0}^2 V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\rho], \quad (5-7b)$$

where $\delta^1 + \mathcal{D}, \delta^2 + \mathcal{D} \in \mathcal{D}^\perp / \mathcal{D}$, $0 \neq \lambda + \mathcal{C} \in \mathcal{C}_{\equiv \tau}^\perp \bmod \mathcal{C}$.

Proof. (i) Similar to Prop. 5.4(iii), we quickly have

$$0 = N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta}(\tau^j) \\ V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}}, V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^i)[0] \end{matrix} \right),$$

when (1) $i = j$ and $\delta \neq 0$ or (2) $i \neq j$. Also, by Prop. 5.5, Prop. 5.4 and Prop. 5.6 we have

$$\begin{aligned} N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon] \\ V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}}, V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^i)[0] \end{matrix} \right) &= N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^{2i})[0] \\ V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}}, V_{L_{\mathcal{C} \times (-\delta + \mathcal{D})}}[-\varepsilon] \end{matrix} \right) = 0, \\ N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})}} \\ V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}}, V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^i)[0] \end{matrix} \right) &= N_{\mathcal{C} \times \mathcal{D}}^\tau \left(\begin{matrix} V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^{2i})[0] \\ V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}}, V_{L_{(-\lambda + \mathcal{C}) \times (-\delta + \mathcal{D})}} \end{matrix} \right) = 0. \end{aligned}$$

Therefore

$$V_{L_{(\gamma+\mathbf{C}) \times \mathcal{D}}} \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0] = \bigoplus_{\rho=0}^2 n_{\rho} V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[\rho], \quad (5-8)$$

for some $n_{\rho} \in \mathbb{N}$. Multiply the equation (5-8) by $V_{L_{\mathbf{C} \times \mathcal{D}}}[h]$, $h = 1, 2$, we have

$$(V_{L_{\mathbf{C} \times \mathcal{D}}}[h] \times V_{L_{(\gamma+\mathbf{C}) \times \mathcal{D}}}) \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0] = V_{L_{\mathbf{C} \times \mathcal{D}}}[h] \times \bigoplus_{\rho=0}^2 n_{\rho} V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[\rho].$$

By Prop. 5.4, the left hand side is equal to

$$V_{L_{(\gamma+\mathbf{C}) \times \mathcal{D}}} \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0] = \bigoplus_{\rho=0}^2 n_{\rho} V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[\rho]$$

while the right hand side is $\bigoplus_{\rho=0}^2 n_{\rho} V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[\rho + h]$; thus, we have

$$\bigoplus_{\rho=0}^2 n_{\rho} V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[\rho] = \bigoplus_{\rho=0}^2 n_{\rho} V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[\rho + h],$$

for all $0 \leq h \leq 2$. This gives $n_0 = n_1 = n_2$. Finally, by comparing the quantum dimensions of both sides of (5-8), we have $3(2^{\ell}/|\mathbf{C}|) = (n_0 + n_1 + n_2)(2^{\ell}/|\mathbf{C}|)$ and hence $n_0 = n_1 = n_2 = 1$. This proves (i).

(ii) By Prop. 5.4 we have

$$V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon] = V_{L_{\mathbf{C} \times ((-1)^i \delta^2 + \mathcal{D})}} [(-1)^{i+1} \varepsilon] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0].$$

Therefore,

$$\begin{aligned} & V_{L_{(\gamma+\mathbf{C}) \times (\delta^1 + \mathcal{D})}} \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon] = V_{L_{(\gamma+\mathbf{C}) \times (\delta^1 + \mathcal{D})}} \times (V_{L_{\mathbf{C} \times ((-1)^i \delta^2 + \mathcal{D})}} [(-1)^{i+1} \varepsilon] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0]) \\ &= V_{L_{\mathbf{C} \times ((-1)^i \delta^2 + \mathcal{D})}} [(-1)^{i+1} \varepsilon] \times (V_{L_{(\gamma+\mathbf{C}) \times (\delta^1 + \mathcal{D})}} \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0]) \\ &= V_{L_{\mathbf{C} \times ((-1)^i \delta^2 + \mathcal{D})}} [(-1)^{i+1} \varepsilon] \times \bigoplus_{\rho=0}^2 V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[\rho] \\ &= \bigoplus_{\rho=0}^2 V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, -i(-1)^i \delta^2}(\tau^i)[i(-1)^{i+1} \varepsilon + \rho] = \bigoplus_{\rho=0}^2 V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\rho - \varepsilon] = \bigoplus_{\rho=0}^2 V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\rho]. \end{aligned}$$

This completes the proof. \square

Case (IV): $V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta^1}(\tau)[\varepsilon^1] \times V_{L_{\mathbf{C} \times \mathcal{D}}}^{T, \eta^2}(\tau^2)[\varepsilon^2]$

Proposition 5.9. *We have the fusion rules:*

$$(i) \quad V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^2)[0] = V_{L_{\mathcal{C} \times \mathcal{D}}}[0] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}} V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}} \quad (5-9a)$$

$$(ii) \quad V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^1}(\tau)[\varepsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^2}(\tau^2)[\varepsilon^2] = V_{L_{\mathcal{C} \times (\eta^2 - \eta^1 + \mathcal{D})}}[\varepsilon^1 - \varepsilon^2] \\ \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}} V_{L_{(\gamma + \mathcal{C}) \times (\eta^2 - \eta^1 + \mathcal{D})}}. \quad (5-9b)$$

In particular, if \mathcal{C} is self-dual, we have

$$V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^1}(\tau)[\varepsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^2}(\tau^2)[\varepsilon^2] = V_{L_{\mathcal{C} \times (\eta^2 - \eta^1 + \mathcal{D})}}[\varepsilon^1 - \varepsilon^2].$$

Proof. (i) By Prop. 5.5 we have

$$N_{\mathcal{C} \times \mathcal{D}}^{\tau} \left(\begin{array}{c} V_{L_{\mathcal{C} \times \mathcal{D}}}[\varepsilon] \\ V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0], V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^2)[0] \end{array} \right) = N_{\mathcal{C} \times \mathcal{D}}^{\tau} \left(\begin{array}{c} V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0] \\ V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0], V_{L_{\mathcal{C} \times \mathcal{D}}}[2\varepsilon] \end{array} \right).$$

By Prop. 5.4 we have

$$N_{\mathcal{C} \times \mathcal{D}}^{\tau} \left(\begin{array}{c} V_{L_{\mathcal{C} \times \mathcal{D}}}[\varepsilon] \\ V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0], V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^2)[0] \end{array} \right) = \begin{cases} 1 & \text{if } \varepsilon = 0; \\ 0 & \text{if } \varepsilon = 1, 2. \end{cases}$$

Similarly, by Prop. 5.5 and Prop. 5.8 we have

$$N_{\mathcal{C} \times \mathcal{D}}^{\tau} \left(\begin{array}{c} V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}} \\ V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0], V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^2)[0] \end{array} \right) = N_{\mathcal{C} \times \mathcal{D}}^{\tau} \left(\begin{array}{c} V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0] \\ V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0], V_{L_{(-\gamma + \mathcal{C}) \times \mathcal{D}}} \end{array} \right) = 1.$$

Therefore,

$$V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^2)[0] \geq V_{L_{\mathcal{C} \times \mathcal{D}}}[0] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}} V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}}.$$

Recall that

$$\text{qdim} (V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^2)[0]) = \left(\frac{2^\ell}{|\mathcal{C}|} \right)^2 = |\mathcal{C}^{\perp} / \mathcal{C}|;$$

$$\text{qdim} V_{L_{\mathcal{C} \times \mathcal{D}}}[0] = 1.$$

Moreover,

$$\text{qdim} \left(\bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}} V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}} \right) = \#\{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}\} \cdot 3.$$

Since

$$\#\{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}\} = \frac{1}{3} (|\mathcal{C}^{\perp} / \mathcal{C}| - 1),$$

we know

$$\begin{aligned} & \text{qdim} \left(V_{L_{\mathcal{C} \times \mathcal{D}}} [0] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}} V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}} \right) = |\mathcal{C}^{\perp} / \mathcal{C}| \\ & = \text{qdim} (V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[0] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^2)[0]). \end{aligned}$$

This proves (i).

(ii) By Prop. 5.4 we have

$$V_{L_{\mathcal{C} \times (\gamma^i + \mathcal{D})}}^{T, \eta^i}(\tau^i)[\varepsilon^i] = V_{L_{\mathcal{C} \times ((-1)^i \eta^i + \mathcal{D})}} [(-1)^{i+1} \varepsilon^i] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0].$$

Therefore,

$$\begin{aligned} & V_{L_{\mathcal{C} \times (\gamma^1 + \mathcal{D})}}^{T, \eta^1}(\tau)[\varepsilon^1] \times V_{L_{\mathcal{C} \times (\gamma^2 + \mathcal{D})}}^{T, \eta^2}(\tau^2)[\varepsilon^2] \\ & = V_{L_{\mathcal{C} \times (-\eta^1 + \mathcal{D})}} [\varepsilon^1] \times V_{L_{\mathcal{C} \times (\eta^2 + \mathcal{D})}} [-\varepsilon^2] \times V_{L_{\mathcal{C} \times \mathbf{0}}}^{T, \mathbf{0}}(\tau)[0] \times V_{L_{\mathcal{C} \times \mathbf{0}}}^{T, \mathbf{0}}(\tau^2)[0] \\ & = V_{L_{\mathcal{C} \times (\eta^2 - \eta^1 + \mathcal{D})}} [\varepsilon^1 - \varepsilon^2] \times \left(V_{L_{\mathcal{C} \times \mathbf{0}}} [0] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}} V_{L_{(\gamma + \mathcal{C}) \times \mathcal{D}}} \right) \\ & = V_{L_{\mathcal{C} \times (\eta^2 - \eta^1 + \mathcal{D})}} [\varepsilon^1 - \varepsilon^2] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}} V_{L_{(\eta^2 - \eta^1 + \gamma + \mathcal{C}) \times \mathcal{D}}}. \end{aligned}$$

This proves (ii). □

Case (V): $V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^1}(\tau^i)[\varepsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^2}(\tau^i)[\varepsilon^2]$

We first consider the case $\delta^1 = \delta^2 = \mathbf{0}$ and $\varepsilon^1 = \varepsilon^2 = 0$. By the similar analysis as in the previous few cases, we can show quickly that many fusion coefficients are zero. Assume

$$\begin{aligned} & V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0] \\ & = x V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^{2i})[0] + y V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^{2i})[1] + z V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^{2i})[2], \end{aligned} \tag{5-10}$$

for some $x, y, z \in \mathbb{Z}_{\geq 0}$.

For simplicity, we denote

$$T[j] := V_{L_{\mathcal{C} \times \mathbf{0}}}^{T, \mathbf{0}}(\tau)[j]; \quad \check{T}[j] := V_{L_{\mathcal{C} \times \mathbf{0}}}^{T, \mathbf{0}}(\tau^2)[j]; \quad S[j] := V_{L_{\mathcal{C} \times \mathcal{D}}} [j].$$

Equation (5-10) then becomes

$$T[0] \times T[0] = x \check{T}[0] + y \check{T}[1] + z \check{T}[2] = \check{T}[0] \times (x S[0] + y S[2] + z S[1]).$$

We multiply this equation by $\check{T}[0]$ and get

$$\check{T}[0] \times T[0] \times T[0] = \check{T}[0] \times \check{T}[0] \times (x S[0] + y S[2] + z S[1]). \tag{5-11}$$

On the left hand side of (5-11), by Prop. 5.4 and Prop. 5.8, we have

$$\begin{aligned} \check{T}[0] \times T[0] \times T[0] &= (\check{T}[0] \times T[0]) \times T[0] = \left(S[0] \oplus \bigoplus_{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}} V_{L_{(\gamma+\mathcal{C}) \times \mathcal{D}}} \right) \times T[0] \\ &= T[0] + Q(T[0] + T[1] + T[2]), \end{aligned} \quad (5-12)$$

where $Q = \#\{\mathbf{0} \neq \gamma \in \mathcal{C}_{\equiv \tau}^{\perp} \bmod \mathcal{C}\} = \frac{4^{\ell-2d}-1}{3}$ and $d = \dim \mathcal{C}$.

By symmetry between automorphisms τ and τ^2 , we can rewrite (5-10) as

$$\check{T}[0] \times \check{T}[0] = xT[0] + yT[1] + zT[2].$$

Therefore, the right hand side of (5-11) becomes

$$\begin{aligned} &(\check{T}[0] \times \check{T}[0]) \times (xS[0] + yS[2] + zS[1]) \\ &= (xT[0] + yT[1] + zT[2]) \times (xS[0] + yS[2] + zS[1]) \\ &= (x^2 + y^2 + z^2)T[0] + (xy + yz + zx)T[1] + (xy + yz + zx)T[2]. \end{aligned} \quad (5-13)$$

Comparing (5-12) and (5-13) we have

$$x + y + z = \text{qdim } T[0] = 2^{\ell-2d}, \quad (5-14a)$$

$$xy + yz + zx = \frac{4^{\ell-2d} - 1}{3}; \quad (5-14b)$$

$$x^2 + y^2 + z^2 = xy + yz + zx + 1 \quad (5-14c)$$

From (5-14c) we know

$$(x - y)^2 + (y - z)^2 + (z - x)^2 = 2.$$

Assuming $x \geq y \geq z \geq 0$, then we have $x - z = 1$ and thus $z + 1 \geq y \geq z$. Therefore,

$$\begin{aligned} x = y &= \frac{2^{\ell-2d} + 1}{3}, & z &= \frac{2^{\ell-2d} - 2}{3}, & \text{if } 2^{\ell-2d} &\equiv 2 \pmod{3}; \\ x &= \frac{2^{\ell-2d} + 2}{3}, & y = z &= \frac{2^{\ell-2d} - 1}{3}, & \text{if } 2^{\ell-2d} &\equiv 1 \pmod{3}, \end{aligned}$$

where $d = \dim \mathcal{C}$. Note that $2^{\ell-2d} \equiv 2^{\ell} \pmod{3}$. As a summary, we have the proposition.

Proposition 5.10. *We have fusion rules:*

$$\begin{aligned} &V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^i)[0] \\ &= x' V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^{2i})[0] + y' V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^{2i})[1] + z' V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^{2i})[2], \end{aligned} \quad (5-15)$$

for some $x', y', z' \in \mathbb{N}$.

If $(x \geq y \geq z)$ is the decreasing reordering of (x', y', z') , then we have

$$\begin{aligned} x = y &= \frac{2^{\ell-2d} + 1}{3}, & z &= \frac{2^{\ell-2d} - 2}{3}, & \text{if } \ell \text{ is odd}; \\ x &= \frac{2^{\ell-2d} + 2}{3}, & y = z &= \frac{2^{\ell-2d} - 1}{3}, & \text{if } \ell \text{ is even}, \end{aligned}$$

where $d = \dim \mathcal{C}$. Note that $2^{\ell-2d} = \sqrt{|\mathcal{C}^\perp / \mathcal{C}|}$.

S-matrix and Verlinde formula. Next we compute the exact fusion rules using Verlinde formula and S -matrix. First we review Dong-Li-Mason's theory on trace functions [DLM00].

Let V be a rational VOA and $g, h \in \text{Aut } V$ be commuting automorphisms of finite orders. Let M be a g -twisted h -stable V -module. There exists a linear isomorphism $\varphi(h)$ of M such that

$$\varphi(h)Y_M(u, z) = Y_M(hu, z)\varphi(h).$$

For a homogeneous $v \in V$ with $L(1)v = 0$ we define the trace function

$$T_M(v, g, h; z) := \text{tr}_M \varphi(h)o(v)q^{L(0)-c/24} = q^{\lambda-c/24} \sum_{n \in \frac{1}{|g|}\mathbb{Z}_+} \text{tr}_{M_{\lambda+n}} o(v)\varphi(h)q^n,$$

where $o(v)$ is the degree zero operator of v , λ is the conformal weight of M , c is the central charge of V and $q = e^{(2\pi\sqrt{-1}z)}$.

Proposition 5.11. [DLM00] *Let $C_1(g, h)$ be the \mathbb{C} -vector space*

$$C_1(g, h) := \text{Span}_{\mathbb{C}}\{T_M(v, g, h; z) \mid M \text{ is a } g\text{-twisted } h\text{-stable } V\text{-module}\}.$$

Then

(i) $C_1(g, h)$ has a basis:

$$\{T_M(v, g, h; z) \mid M \text{ is an irreducible } g\text{-twisted } h\text{-stable } V\text{-module}\}.$$

(ii) *Modular invariance: Let $T_M(v, g, h; z) \in C_1(g, h)$ and $\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then we have $T_M(v, g, h; \Gamma \circ z) \in C_1(g, h) \circ \Gamma$ in the sense that*

$$T_M(v, g, h; \frac{az+b}{cz+d}) \in C_1(g^a h^c, g^b h^d).$$

In fact, if M is a g -twisted h -stable V -module, then

$$T_M(v, g, h; \frac{az+b}{cz+d}) = \sum S_N^{(g,h)} T_N(v, g, h; z),$$

where N runs over irreducible $g^a h^c$ -twisted $g^b h^d$ -stable V -module, and the coefficients $S_N^{(g,h)}$ are independent of v .

In particular, when $g = h = \text{id}$, $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $V = M^0, \dots, M^d$ are all inequivalent irreducible V -modules, we have

$$T_{M^i}(v, \text{id}, \text{id}; -\frac{1}{z}) = \sum_{j=0}^d S_{i,j} T_{M^j}(z). \quad (5-16)$$

For simplicity, we denote

$$M(g, h; z) = Z_M(g, h; z) := T_M(\mathbb{1}, g, h; z),$$

and

$$M(z) := Z_M(\text{id}, \text{id}; z) = \text{ch } M(z).$$

Definition 5.12. The matrix $S = (S_{i,j})$ defined in equation (5-16) is called the S -matrix.

Theorem 5.13. [Hua08] *Let V be a rational and C_2 -cofinite simple VOA of CFT type and assume $V \cong V'$. Let $S = (S_{i,j})_{i,j=0}^d$ be the S -matrix as defined in (5-16). Then*

- (i) $(S^{-1})_{i,j} = S_{i,j'} = S_{i',j}$, and $S_{i',j'} = S_{i,j}$;
- (ii) S is symmetric and $S^2 = (\delta_{i,j'})$;
- (iii) $N_{i,j}^k = \sum_{s=0}^d \frac{S_{j,s} S_{i,s} S_{s,k}^{-1}}{S_{0,s}}$;
- (iv) The S -matrix diagonalizes the fusion matrix $N(i) = (N_{i,j}^k)_{j,k=0}^d$ with diagonal entries $\frac{S_{i,s}}{S_{0,s}}$, for $i, s = 0, \dots, d$. More explicitly, $S^{-1}N(i)S = \text{diag}(\frac{S_{i,s}}{S_{0,s}})_{s=0}^d$. In particular, $S_{0,s} \neq 0$ for $s = 0, \dots, d$.

Proposition 5.14. [DJX13] *Let V be a simple, rational and C_2 -cofinite VOA of CFT type. Let M^0, M^1, \dots, M^d be as before with the corresponding conformal weights $\lambda_i > 0$ for $0 < i \leq d$. Then $0 < \text{qdim}_V M^i < \infty$ for any $0 \leq i \leq d$. In fact, we have*

$$\text{qdim}_V M^i = \frac{S_{i,0}}{S_{0,0}}. \quad (5-17)$$

The case that \mathcal{D} is self-dual. Denote $\xi = e^{(2\pi\sqrt{-1})/3}$ a primitive cubic root of unity.

We define a function $\Xi : \mathbb{Z} \rightarrow \{-1, 2\}$ by

$$\Xi(n) := \xi^n + \xi^{2n}, \quad \text{for } n \in \mathbb{Z}.$$

Note that

$$\Xi(n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{3}, \\ -1 & \text{otherwise.} \end{cases}$$

Proposition 5.15. *Let \mathcal{D} be a self-dual \mathbb{Z}_3 -code of length ℓ , then we have*

$$\begin{aligned} T[0] \times T[0] &= \frac{2^{\ell-2d} + \Xi(\ell)}{3} \check{T}[0] + \frac{2^{\ell-2d} + \Xi(\ell+2)}{3} \check{T}[1] + \frac{2^{\ell-2d} + \Xi(\ell+1)}{3} \check{T}[2] \\ &= \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d} + \Xi(\ell-\varepsilon)}{3} \check{T}[\varepsilon]. \end{aligned}$$

Proof. We mimic the proof of [Miy13, Lemma 18].

Let V denote the lattice VOA $V_{L_{\mathbf{C} \times \mathbf{D}}}$. Since \mathbf{D} is self-dual, we know V is the unique τ -stable irreducible module. Thus, V has exactly one τ^i -twisted module for each $i = 1, 2$. We denoted them by T and \check{T} , respectively. Let

$$M^i := V[i], \quad M^{3+i} := T[i], \quad M^{6+i} := \check{T}[i],$$

for $i = 0, 1, 2$. Then we know M^j , ($j = 0, \dots, 8$) are irreducible V^τ -modules. Note that there are also irreducible V^τ -modules which are not τ -stable, but we won't need in the proof.

Denote $C_1(g, h)$ the vector space generated by trace functions of g -twisted and h -stable V -modules. By [DLM00] we know the modular transformation $\Gamma : z \mapsto \frac{-1}{z}$ maps $C_1(g, h)$ to $C_1(h, g^{-1})$. In particular, Γ sends $C_1(\tau, \tau^j)$ to $C_1(\tau^j, \tau^2)$ for $j = 0, 1, 2$.

First, we know $Z_T(\tau, 1; \frac{-1}{z}) \in C_1(1, \tau^2)$ which is spanned by $Z_V(1, 1; z)$. Therefore, we can write

$$Z_T(\tau, 1; \frac{-1}{z}) = \lambda_1 Z_V(1, 1; z),$$

for some $\lambda_1 \in \mathbb{C}$.

Denote $W^i(g, h, z) = Z_{M^i}(g, h; z)$ for any i . Then we have

$$W^3(\frac{-1}{z}) + W^4(\frac{-1}{z}) + W^5(\frac{-1}{z}) = \lambda_1 (W^0(z) + W^1(z) + W^2(z)). \quad (5-18)$$

Similarly, using $Z_T(\tau, \tau^j; \frac{-1}{z}) \in C_1(\tau^j, \tau^2)$ for $j = 1, 2$, we can write

$$W^3(1, \tau, \frac{-1}{z}) + W^4(1, \tau, \frac{-1}{z}) + W^5(1, \tau, \frac{-1}{z}) = \mu_1 \left(W^3(1, \tau^2; z) + W^4(1, \tau^2; z) + W^5(1, \tau^2; z) \right),$$

and

$$W^3(1, \tau^2, \frac{-1}{z}) + W^4(1, \tau^2, \frac{-1}{z}) + W^5(1, \tau^2, \frac{-1}{z}) = \mu_2 \left(W^6(1, \tau^2; z) + W^7(1, \tau^2; z) + W^8(1, \tau^2; z) \right), \quad (5-19)$$

for some $\mu_1, \mu_2 \in \mathbb{C}$.

We can define a linear isomorphism $\varphi(\tau^j)$ as following:

$$\varphi(\tau^j) = \xi^{ij} \text{ on } M^{3+i} \text{ and } M^{6+i}.$$

Therefore we can rewrite the equation (5-19) as

$$\begin{aligned} W^3(\tau, 1; \frac{-1}{z}) + \xi W^4(\tau, 1; \frac{-1}{z}) + \xi^2 W^5(\tau, 1; \frac{-1}{z}) &= \mu_1 \left(W^3(\tau, 1; z) + \xi^2 W^4(\tau, 1; z) + \xi W^5(\tau, 1; z) \right), \\ W^3(\tau, 1; \frac{-1}{z}) + \xi^2 W^4(\tau, 1; \frac{-1}{z}) + \xi W^5(\tau, 1; \frac{-1}{z}) &= \mu_2 \left(W^6(\tau^2, 1; z) + \xi^2 W^7(\tau^2, 1; z) + \xi W^8(\tau^2, 1; z) \right). \end{aligned} \quad (5-20)$$

Solving equations (5-18) and (5-20), we know

$$\begin{aligned}
W^3\left(\frac{-1}{z}\right) &= \frac{\lambda_1}{3}(W^0(z) + \xi^2 W^1(z) + \xi W^2(z)) \\
&\quad + \frac{\mu_1}{3}(W^3(z) + \xi^2 W^4(z) + \xi W^5(z)) + \frac{\mu_2}{3}(W^6(z) + \xi W^7(z) + \xi^2 W^8(z)), \\
W^4\left(\frac{-1}{z}\right) &= \frac{\lambda_1}{3}(W^0(z) + \xi^2 W^1(z) + \xi W^2(z)) \\
&\quad + \frac{\mu_1}{3}(\xi^2 W^3(z) + \xi W^4(z) + W^5(z)) + \frac{\mu_2}{3}(\xi W^6(z) + \xi^2 W^7(z) + W^8(z)), \\
W^5\left(\frac{-1}{z}\right) &= \frac{\lambda_1}{3}(W^0(z) + \xi^2 W^1(z) + \xi W^2(z)) \\
&\quad + \frac{\mu_1}{3}(\xi W^3(z) + W^4(z) + \xi^2 W^5(z)) + \frac{\mu_2}{3}(\xi^2 W^6(z) + W^7(z) + \xi W^8(z)).
\end{aligned}$$

In other words, the rows $S_{i,j}$ for $i = 3, 4, 5$ are given by

$$\frac{1}{3} \begin{pmatrix} \lambda_1 & \xi^2 \lambda_1 & \xi \lambda_1 & \mu_1 & \xi^2 \mu_1 & \xi \mu_1 & \mu_2 & \xi \mu_2 & \xi^2 \mu_2 & 0 & \cdots & 0 \\ \lambda_1 & \xi^2 \lambda_1 & \xi \lambda_1 & \xi^2 \mu_1 & \xi \mu_1 & \mu_1 & \xi \mu_2 & \xi^2 \mu_2 & \mu_2 & 0 & \cdots & 0 \\ \lambda_1 & \xi^2 \lambda_1 & \xi \lambda_1 & \xi \mu_1 & \mu_1 & \xi^2 \mu_1 & \xi^2 \mu_2 & \mu_2 & \xi \mu_2 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $S_{0,0}^2 \text{glob}(V^\tau) = 1$, we know

$$\begin{aligned}
S_{0,0}^2 \cdot 9 |\mathcal{C}^\perp / \mathcal{C}| |\mathcal{D}^\perp / \mathcal{D}| &= 1, \\
S_{0,i} / S_{0,0} &= \text{qdim } M^i,
\end{aligned}$$

and

$$\text{qdim } M^i = \begin{cases} 1, & \text{if } i = 0, 1, 2 \\ \frac{2^\ell}{|\mathcal{C}|}, & \text{if } 3 \leq i \leq 8. \end{cases}$$

This gives $S_{0,0} = S_{0,1} = S_{0,2} = \frac{\pm 2^{2d-\ell}}{3}$ and $\lambda_1 = 3S_{0,h} = \pm 1$ for $3 \leq h \leq 8$.

By Verlinde formula, we know fusion rules are given by

$$\begin{aligned}
N_{3,3}^6 &= \frac{3 \cdot (\lambda_1/3)^3}{S_{0,0}} + \frac{3((\mu_1/3)^3 + (\mu_2/3)^3)}{(\lambda_1/3)} = \frac{\pm(2^{\ell-2d} + \mu_1^3 + \mu_2^3)}{3}, \\
N_{3,3}^7 &= \frac{\pm(2^{\ell-2d} + \xi^2 \mu_1^3 + \xi \mu_2^3)}{3}, \\
N_{3,3}^8 &= \frac{\pm(2^{\ell-2d} + \xi \mu_1^3 + \xi^2 \mu_2^3)}{3}.
\end{aligned}$$

Since

$$N_{3,3}^6 + N_{3,3}^7 + N_{3,3}^8 = 2^{\ell-2d},$$

we know

$$S_{0,0} = 2^{2d-\ell},$$

and

$$\begin{aligned} N_{3,3}^6 &= \frac{2^{\ell-2d} + \mu_1^3 + \mu_2^3}{3}, \\ N_{3,3}^7 &= \frac{2^{\ell-2d} + \xi^2 \mu_1^3 + \xi \mu_2^3}{3}, \\ N_{3,3}^8 &= \frac{2^{\ell-2d} + \xi \mu_1^3 + \xi^2 \mu_2^3}{3}. \end{aligned}$$

By $S_{3,6} = 1$ we have

$$9 = 3\lambda_1^2 + 6\mu_1\mu_2,$$

and hence

$$\mu_1\mu_2 = 1.$$

Notice that the weights of irreducible V^τ -modules of twisted type are

$$\text{wt } V_{L_{\mathbf{C} \times \mathcal{D}}}^{T,0}(\tau^i)[\varepsilon] \in \ell/9 + 1/3\left(\sum_{\mathbf{e} \in S_\varepsilon} e_i\right) + \mathbb{Z} = \varepsilon/3 + \ell/9 + \mathbb{Z},$$

for $i = 1, 2$. By considering the characters, we have from the above S -matrix that

$$\begin{aligned} Z_V(1, \tau; z) &= \text{ch}(W^0) + \xi \text{ch}(W^1) + \xi^2 \text{ch}(W^2), \\ Z_V(1, \tau; -1/z) &= \lambda_1 \{ \text{ch}(W^3) + \text{ch}(W^4) + \text{ch}(W^5) \}, \\ Z_V(1, \tau; -1/(z+1)) &= e^{2\pi\sqrt{-1}N/24} \cdot e^{2\pi\sqrt{-1}\ell/9} \lambda_1 \{ \text{ch}(W^3) + \xi \text{ch}(W^4) + \xi^2 \text{ch}(W^5) \}, \\ Z_V(1, \tau; -1/((-1/z)+1)) &= e^{2\pi\sqrt{-1}N/24} \cdot e^{2\pi\sqrt{-1}\ell/9} \lambda_1 \mu_1 \{ \text{ch}(W^3) + \xi^2 \text{ch}(W^4) + \xi \text{ch}(W^5) \}, \end{aligned}$$

where $N = 2\ell$ is the rank of the lattice $L_{\mathbf{C} \times \mathcal{D}}$. On the other hand, since

$$\begin{aligned} Z_V(1, \tau; -1/((-1/z)+1)) &= Z_V(1, \tau; -1 - \frac{1}{z-1}) \\ &= e^{-2\pi\sqrt{-1}N/24} Z_V(1, \tau, -1/(z-1)) \\ &= e^{-4\pi\sqrt{-1}N/24} \cdot e^{-2\pi\sqrt{-1}\ell/9} \lambda_1 \{ \text{ch}(W^3) + \xi^2 \text{ch}(W^4) + \xi \text{ch}(W^5) \} \end{aligned}$$

we have $\mu_1 \cdot e^{6\pi\sqrt{-1}N/24} \cdot e^{4\pi\sqrt{-1}\ell/9} = 1$. Since $N = 2\ell$ and ℓ is a multiple of 4, we know $8|N$ and $\mu_1 = e^{-4\pi\sqrt{-1}\ell/9}$. Using $\mu_1\mu_2 = 1$, we have $\mu_1^3 = \xi^\ell$ and $\mu_2^3 = \xi^{2\ell}$. This gives

$$\begin{aligned} &T[0] \times T[0] \\ &= \frac{2^{\ell-2d} + \xi^\ell + \xi^{2\ell}}{3} \check{T}[0] + \frac{2^{\ell-2d} + \xi^{\ell+2} + \xi^{2\ell+1}}{3} \check{T}[1] + \frac{2^{\ell-2d} + \xi^{\ell+1} + \xi^{2\ell+2}}{3} \check{T}[2] \\ &= \frac{2^{\ell-2d} + \Xi(\ell)}{3} \check{T}[0] + \frac{2^{\ell-2d} + \Xi(\ell+2)}{3} \check{T}[1] + \frac{2^{\ell-2d} + \Xi(\ell+1)}{3} \check{T}[2], \end{aligned}$$

and completes the proof. \square

General Case. Recall the decomposition

$$V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta}(\tau^i)[\varepsilon] \cong \bigoplus_{\gamma \in \mathcal{D}} V_{L_{\mathcal{C} \times \mathbf{0}}}^{T, \eta - i\gamma}(\tau^i)[\varepsilon]. \quad (5-21)$$

In the following, we will denote

$$\begin{aligned} T_{\mathcal{C} \times \mathcal{D}}^{\eta}[\varepsilon] &:= V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta}(\tau)[\varepsilon], \\ \check{T}_{\mathcal{C} \times \mathcal{D}}^{\eta}[\varepsilon] &:= V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta}(\tau^2)[\varepsilon], \end{aligned}$$

in addition, we let

$$\begin{aligned} T_{\mathcal{C} \times \mathcal{D}}[\varepsilon] &:= V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau)[\varepsilon], \\ \check{T}_{\mathcal{C} \times \mathcal{D}}[\varepsilon] &:= V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \mathbf{0}}(\tau^2)[\varepsilon]. \end{aligned}$$

We also let $\mathbf{0}$ be the trivial \mathbb{Z}_3 -code of various length depending on context.

Proposition 5.16. *Let \mathcal{B} be a self-dual \mathbb{F}_4 -code of length 2. Then*

$$T_{\mathcal{B} \times \mathbf{0}}[0] \times T_{\mathcal{B} \times \mathbf{0}}[0] = \check{T}_{\mathcal{B} \times \mathbf{0}}[2].$$

Proof. In this case, all irreducible modules are simple current. It suffices to find the nonzero fusion rules.

Let $\mathcal{B}^2 := \mathcal{B} \oplus \mathcal{B}$ be a self-dual code of length 4 and let \mathcal{S} be a self-dual \mathbb{Z}_3 -code of length 4. By Prop. 5.15, we know

$$T_{\mathcal{B}^2 \times \mathcal{S}}[0] \times T_{\mathcal{B}^2 \times \mathcal{S}}[0] = \check{T}_{\mathcal{B}^2 \times \mathcal{S}}[1].$$

Considering the sub VOA $V_{L_{\mathcal{B}^2 \times \mathbf{0}}}^{\tau} \subset V_{L_{\mathcal{B}^2 \times \mathcal{S}}}^{\tau}$, we have the decomposition of $V_{L_{\mathcal{B}^2 \times \mathbf{0}}}^{\tau}$ -modules

$$\begin{aligned} T_{\mathcal{B}^2 \times \mathcal{S}}[\varepsilon] &= \bigoplus_{\eta \in \mathcal{S}} T_{\mathcal{B}^2 \times \mathbf{0}}^{\eta}[\varepsilon], \\ \check{T}_{\mathcal{B}^2 \times \mathcal{S}}[\varepsilon] &= \bigoplus_{\eta \in \mathcal{S}} \check{T}_{\mathcal{B}^2 \times \mathbf{0}}^{\eta}[\varepsilon]. \end{aligned}$$

By Prop. 5.3 we have

$$1 = N \left(\begin{matrix} \check{T}_{\mathcal{B}^2 \times \mathcal{S}}[1] \\ T_{\mathcal{B}^2 \times \mathcal{S}}[0], T_{\mathcal{B}^2 \times \mathcal{S}}[0] \end{matrix} \right) \leq N \left(\begin{matrix} \bigoplus_{\eta \in \mathcal{S}} \check{T}_{\mathcal{B}^2 \times \mathbf{0}}^{\eta}[1] \\ T_{\mathcal{B}^2 \times \mathbf{0}}[0], T_{\mathcal{B}^2 \times \mathbf{0}}[0] \end{matrix} \right) = N \left(\begin{matrix} \check{T}_{\mathcal{B}^2 \times \mathbf{0}}[1] \\ T_{\mathcal{B}^2 \times \mathbf{0}}[0], T_{\mathcal{B}^2 \times \mathbf{0}}[0] \end{matrix} \right),$$

where the last equality follows from Prop. 5.10. Since \mathcal{B}^2 is self-dual, then we have

$$T_{\mathcal{B}^2 \times \mathbf{0}}[0] \times T_{\mathcal{B}^2 \times \mathbf{0}}[0] = \check{T}_{\mathcal{B}^2 \times \mathbf{0}}[1].$$

Now consider the subVOA $V_{\mathcal{B} \times \mathbf{0}}^{\tau} \otimes V_{\mathcal{B} \times \mathbf{0}}^{\tau} \subset V_{L_{\mathcal{B}^2 \times \mathbf{0}}}^{\tau}$ and the decomposition

$$T_{\mathcal{B}^2 \times \mathbf{0}}[\varepsilon] = \bigoplus_{\varepsilon_0=0,1,2} T_{\mathcal{B} \times \mathbf{0}}[\varepsilon_0] \otimes T_{\mathcal{B} \times \mathbf{0}}[\varepsilon - \varepsilon_0]$$

of $V_{\mathcal{B} \times \mathbf{0}}^{\tau} \otimes V_{\mathcal{B} \times \mathbf{0}}^{\tau}$ -modules.

Using the decomposition we have

$$\begin{aligned} 1 &= N\left(\begin{matrix} \check{T}_{\mathbf{B}^2 \times \mathbf{0}}^\eta[1] \\ T_{\mathbf{B}^2 \times \mathbf{0}}[0], T_{\mathbf{B}^2 \times \mathbf{0}}[0] \end{matrix}\right) \leq N\left(\begin{matrix} \sum_{\varepsilon_0=0,1,2} \check{T}_{\mathbf{B} \times \mathbf{0}}[\varepsilon_0] \otimes \check{T}_{\mathbf{B} \times \mathbf{0}}[1 - \varepsilon_0] \\ T_{\mathbf{B} \times \mathbf{0}}[0] \otimes T_{\mathbf{B} \times \mathbf{0}}[0], T_{\mathbf{B} \times \mathbf{0}}[0] \otimes T_{\mathbf{B} \times \mathbf{0}}[0] \end{matrix}\right) \\ &= \sum_{\varepsilon_0=0,1,2} N\left(\begin{matrix} \check{T}_{\mathbf{B} \times \mathbf{0}}[\varepsilon_0] \\ T_{\mathbf{B} \times \mathbf{0}}[0], T_{\mathbf{B} \times \mathbf{0}}[0] \end{matrix}\right) N\left(\begin{matrix} \check{T}_{\mathbf{B} \times \mathbf{0}}[1 - \varepsilon_0] \\ T_{\mathbf{B} \times \mathbf{0}}[0], T_{\mathbf{B} \times \mathbf{0}}[0] \end{matrix}\right). \end{aligned} \quad (5-22)$$

Now since \mathbf{B} is self-dual, only one of the fusion rules $N\left(\begin{matrix} \check{T}_{\mathbf{B} \times \mathbf{0}}[\varepsilon_0] \\ T_{\mathbf{B} \times \mathbf{0}}[0], T_{\mathbf{B} \times \mathbf{0}}[0] \end{matrix}\right)$, $(\varepsilon = 0, 1, 2)$ is nonzero. To have the above inequality (5-22), we must have $N\left(\begin{matrix} \check{T}_{\mathbf{B} \times \mathbf{0}}[\varepsilon_0] \\ T_{\mathbf{B} \times \mathbf{0}}[0], T_{\mathbf{B} \times \mathbf{0}}[0] \end{matrix}\right) = \delta_{\varepsilon_0, 2}$. This completes the proof. \square

Proposition 5.17. *Let \mathbf{C} and \mathbf{D} be self-orthogonal codes of length ℓ .*

(i) *If the length ℓ is even, then we have*

$$T[0] \times T[0] = \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d} + \Xi(\ell - \varepsilon)}{3} \check{T}[\varepsilon].$$

(ii) *If the length ℓ is odd, then we have*

$$T[0] \times T[0] = \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d} - \Xi(\ell - \varepsilon)}{3} \check{T}[\varepsilon].$$

As a summary, we have

$$T[0] \times T[0] = \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d} + (-1)^\ell \Xi(\ell - \varepsilon)}{3} \check{T}[\varepsilon].$$

It also implies

$$V_{L_{\mathbf{C} \times \mathbf{D}}}^{T, \eta_1}(\tau^i)[\varepsilon_1] \times V_{L_{\mathbf{C} \times \mathbf{D}}}^{T, \eta_2}(\tau^i)[\varepsilon_2] = \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d} + (-1)^\ell \Xi(\ell - \varepsilon)}{3} V_{L_{\mathbf{C} \times \mathbf{D}}}^{T, -(\eta_1 + \eta_2)}(\tau^{2i})[\varepsilon - \varepsilon_1 - \varepsilon_2].$$

Proof. (i) First we assume ℓ is a multiple of 4. Then there exists a self-dual code \mathbf{S} of length ℓ .

Restricting to $V_{L_{\mathbf{C} \times \mathbf{0}}}^\tau$ -modules, we know

$$N\left(\begin{matrix} \check{T}_{\mathbf{C} \times \mathbf{S}}[\varepsilon] \\ T_{\mathbf{C} \times \mathbf{S}}[0], T_{\mathbf{C} \times \mathbf{S}}[0] \end{matrix}\right) \leq N\left(\begin{matrix} \oplus_{\eta \in \mathbf{S}} \check{T}_{\mathbf{C} \times \mathbf{0}}^\eta[\varepsilon] \\ T_{\mathbf{C} \times \mathbf{0}}[0], T_{\mathbf{C} \times \mathbf{0}}[0] \end{matrix}\right) = N\left(\begin{matrix} \check{T}_{\mathbf{C} \times \mathbf{0}}[\varepsilon] \\ T_{\mathbf{C} \times \mathbf{0}}[0], T_{\mathbf{C} \times \mathbf{0}}[0] \end{matrix}\right). \quad (5-23)$$

On the other hand, we know from (5-14a) that

$$\sum_{\varepsilon=0,1,2} N\left(\begin{matrix} \check{T}_{\mathbf{C} \times \mathbf{S}}[\varepsilon] \\ T_{\mathbf{C} \times \mathbf{S}}[0], T_{\mathbf{C} \times \mathbf{S}}[0] \end{matrix}\right) = \sum_{\varepsilon=0,1,2} N\left(\begin{matrix} \check{T}_{\mathbf{C} \times \mathbf{0}}[\varepsilon] \\ T_{\mathbf{C} \times \mathbf{0}}[0], T_{\mathbf{C} \times \mathbf{0}}[0] \end{matrix}\right).$$

Therefore, the inequality in (5-23) must attain equality and we prove (i) when \mathbf{D} is the trivial code of length divisible by 4.

Now let \mathcal{D} be a self-orthogonal code of length ℓ . Similarly, we have

$$N\left(\begin{array}{c} \check{T}_{\mathcal{C} \times \mathcal{D}}[\varepsilon] \\ T_{\mathcal{C} \times \mathcal{D}}[0], T_{\mathcal{C} \times \mathcal{D}}[0] \end{array}\right) \leq N\left(\begin{array}{c} T_{\mathcal{C} \times \mathbf{0}}[\varepsilon] \\ T_{\mathcal{C} \times \mathbf{0}}[0], T_{\mathcal{C} \times \mathbf{0}}[0] \end{array}\right).$$

The same argument as in the case for $\mathcal{D} = \mathbf{0}$ shows

$$N\left(\begin{array}{c} \check{T}_{\mathcal{C} \times \mathcal{D}}[\varepsilon] \\ T_{\mathcal{C} \times \mathcal{D}}[0], T_{\mathcal{C} \times \mathcal{D}}[0] \end{array}\right) = N\left(\begin{array}{c} T_{\mathcal{C} \times \mathbf{0}}[\varepsilon] \\ T_{\mathcal{C} \times \mathbf{0}}[0], T_{\mathcal{C} \times \mathbf{0}}[0] \end{array}\right).$$

This implies

$$N\left(\begin{array}{c} \check{T}_{\mathcal{C} \times \mathcal{D}}[\varepsilon] \\ T_{\mathcal{C} \times \mathcal{D}}[0], T_{\mathcal{C} \times \mathcal{D}}[0] \end{array}\right) = N\left(\begin{array}{c} \check{T}_{\mathcal{C} \times \mathcal{S}}[\varepsilon] \\ T_{\mathcal{C} \times \mathcal{S}}[0], T_{\mathcal{C} \times \mathcal{S}}[0] \end{array}\right),$$

and proves (i) by Prop. 5.15 when ℓ is a multiple of 4.

Now assume $\ell \equiv 2 \pmod{4}$. Let \mathcal{B} be a self-dual \mathbb{F}_4 -code of length 2. Then $\mathcal{C} \oplus \mathcal{B}$ is a self-orthogonal code of length divisible by 4 and $(\mathcal{D} \oplus \mathbf{0})$ is a self-orthogonal code of the same length.

Restricting to

$$V_{L_{\mathcal{C} \times \mathcal{D}}}^T \otimes V_{L_{\mathcal{B} \times \mathbf{0}}}^T \subset V_{L_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}}^T,$$

we know

$$T_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[\varepsilon] = \bigoplus_{\varepsilon_0=0,1,2} T_{\mathcal{C} \times \mathcal{D}}[\varepsilon_0] \otimes T_{\mathcal{B} \times \mathbf{0}}[\varepsilon - \varepsilon_0];$$

moreover

$$\begin{aligned} & N\left(\begin{array}{c} \check{T}_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[\varepsilon] \\ T_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[0], T_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[0] \end{array}\right) \\ & \leq \bigoplus_{\varepsilon_0=0,1,2} N\left(\begin{array}{c} \check{T}_{\mathcal{C} \times \mathcal{D}}[\varepsilon - \varepsilon_0] \\ T_{\mathcal{C} \times \mathcal{D}}[0], T_{\mathcal{C} \times \mathcal{D}}[0] \end{array}\right) N\left(\begin{array}{c} \check{T}_{\mathcal{B} \times \mathbf{0}}[\varepsilon_0] \\ T_{\mathcal{B} \times \mathbf{0}}[0], T_{\mathcal{B} \times \mathbf{0}}[0] \end{array}\right). \end{aligned}$$

By Prop. 5.16 we know

$$T_{\mathcal{B} \times \mathbf{0}}[0] \times T_{\mathcal{B} \times \mathbf{0}}[0] = \check{T}_{\mathcal{B} \times \mathbf{0}}[2];$$

therefore the above inequality becomes

$$N\left(\begin{array}{c} \check{T}_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[\varepsilon] \\ T_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[0], T_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[0] \end{array}\right) \leq N\left(\begin{array}{c} \check{T}_{\mathcal{C} \times \mathcal{D}}[\varepsilon - 2] \\ T_{\mathcal{C} \times \mathcal{D}}[0], T_{\mathcal{C} \times \mathcal{D}}[0] \end{array}\right).$$

We know $\sqrt{\left|\frac{(\mathcal{C} \oplus \mathcal{B})^\perp}{\mathcal{C} \oplus \mathcal{B}}\right|} = \sqrt{\left|\frac{\mathcal{C}^\perp}{\mathcal{C}}\right|}$ and hence by (5-14a)

$$\sum_{\varepsilon=0,1,2} N\left(\begin{array}{c} \check{T}_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[\varepsilon] \\ T_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[0], T_{(\mathcal{C} \oplus \mathcal{B}) \times (\mathcal{D} \oplus \mathbf{0})}[0] \end{array}\right) = \sum_{\varepsilon=0,1,2} N\left(\begin{array}{c} \check{T}_{\mathcal{C} \times \mathcal{D}}[\varepsilon] \\ T_{\mathcal{C} \times \mathcal{D}}[0], T_{\mathcal{C} \times \mathcal{D}}[0] \end{array}\right).$$

Therefore, we must have

$$N\left(\begin{array}{c} \check{T}_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus \mathbf{0})}[\varepsilon] \\ T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus \mathbf{0})}[0], T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus \mathbf{0})}[0] \end{array}\right) = N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon - 2] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right).$$

Note that $\mathbf{C} \oplus \mathbf{B}$ has length $\ell + 2$, thus we have

$$\begin{aligned} N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) &= N\left(\begin{array}{c} \check{T}_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus \mathbf{0})}[\varepsilon + 2] \\ T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus \mathbf{0})}[0], T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus \mathbf{0})}[0] \end{array}\right) \\ &= \frac{2^{\ell-2d} + \Xi(\ell + 2 - \varepsilon - 2)}{3} = \frac{2^{\ell-2d} + \Xi(\ell - \varepsilon)}{3}. \end{aligned}$$

This proves (i) when $\ell \equiv 2 \pmod{4}$.

Now assume ℓ is odd, let $\mathbf{C}_e := \mathbf{C} \oplus \mathbf{0}$ and $\mathbf{D}_e := \mathbf{D} \oplus \mathbf{0}$ be self-orthogonal codes of even length $\ell + 1$. Restricting to the subVOA $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau \otimes V_L^\tau$, we have decomposition of $V_{L_{\mathbf{C} \times \mathbf{D}}}^\tau \otimes V_{\mathbf{0} \times \mathbf{0}}^\tau$ -modules

$$T_{\mathbf{C}_e \times \mathbf{D}_e}[0] = \bigoplus_{\varepsilon_0=0,1,2} T_{\mathbf{C} \times \mathbf{D}}[\varepsilon_0] \otimes T_{\mathbf{0} \times \mathbf{0}}[-\varepsilon_0].$$

We know that $V_{\mathbf{0} \times \mathbf{0}}^\tau = V_L^\tau$. Recall the fusion rules of V_L^τ :

$$T_{\mathbf{0} \times \mathbf{0}}[0] \times T_{\mathbf{0} \times \mathbf{0}}[0] = T_{\mathbf{0} \times \mathbf{0}}[0] + T_{\mathbf{0} \times \mathbf{0}}[2].$$

Therefore,

$$\begin{aligned} N\left(\begin{array}{c} \check{T}_{\mathbf{C}_e \times \mathbf{D}_e}[0] \\ T_{\mathbf{C}_e \times \mathbf{D}_e}[0], T_{\mathbf{C}_e \times \mathbf{D}_e}[0] \end{array}\right) &\leq N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[0] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) + N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[1] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right), \\ N\left(\begin{array}{c} \check{T}_{\mathbf{C}_e \times \mathbf{D}_e}[1] \\ T_{\mathbf{C}_e \times \mathbf{D}_e}[0], T_{\mathbf{C}_e \times \mathbf{D}_e}[0] \end{array}\right) &\leq N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[1] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) + N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[2] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right), \quad (5-24) \\ N\left(\begin{array}{c} \check{T}_{\mathbf{C}_e \times \mathbf{D}_e}[2] \\ T_{\mathbf{C}_e \times \mathbf{D}_e}[0], T_{\mathbf{C}_e \times \mathbf{D}_e}[0] \end{array}\right) &\leq N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[0] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) + N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[2] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right). \end{aligned}$$

This gives

$$\sum_{\varepsilon=0,1,2} N\left(\begin{array}{c} \check{T}_{\mathbf{C}_e \times \mathbf{D}_e}[\varepsilon] \\ T_{\mathbf{C}_e \times \mathbf{D}_e}[0], T_{\mathbf{C}_e \times \mathbf{D}_e}[0] \end{array}\right) \leq 2 \sum_{\varepsilon=0,1,2} N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right).$$

From (5-14a) we know

$$\sum_{\varepsilon=0,1,2} N\left(\begin{array}{c} \check{T}_{\mathbf{C}_e \times \mathbf{D}_e}[\varepsilon] \\ T_{\mathbf{C}_e \times \mathbf{D}_e}[0], T_{\mathbf{C}_e \times \mathbf{D}_e}[0] \end{array}\right) = 2^{\ell+1-2d} = 2 \cdot 2^{\ell-2d} = 2 \sum_{\varepsilon=0,1,2} N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right).$$

Therefore inequalities in (5-24) must attain equalities. This gives

$$\begin{aligned}
N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[0] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) &= 2^{\ell-2d} - \left(N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[1] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) + N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[2] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) \right) \\
&= 2^{\ell-2d} - N\left(\begin{array}{c} \check{T}_{\mathbf{C}_e \times \mathbf{D}_e}[1] \\ T_{\mathbf{C}_e \times \mathbf{D}_e}[0], T_{\mathbf{C}_e \times \mathbf{D}_e}[0] \end{array}\right) = 2^{\ell-2d} - \frac{2^{\ell+1-2d} + \Xi(\ell+1-1)}{3} \\
&= \frac{2^{\ell-2d} - \Xi(\ell)}{3}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[1] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) &= 2^{\ell-2d} - N\left(\begin{array}{c} \check{T}_{\mathbf{C}_e \times \mathbf{D}_e}[2] \\ T_{\mathbf{C}_e \times \mathbf{D}_e}[0], T_{\mathbf{C}_e \times \mathbf{D}_e}[0] \end{array}\right) = \frac{2^{\ell-2d} - \Xi(\ell+1-2)}{3}, \\
N\left(\begin{array}{c} \check{T}_{\mathbf{C} \times \mathbf{D}}[2] \\ T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0] \end{array}\right) &= 2^{\ell-2d} - N\left(\begin{array}{c} \check{T}_{\mathbf{C}_e \times \mathbf{D}_e}[0] \\ T_{\mathbf{C}_e \times \mathbf{D}_e}[0], T_{\mathbf{C}_e \times \mathbf{D}_e}[0] \end{array}\right) = \frac{2^{\ell-2d} - \Xi(\ell+1)}{3}.
\end{aligned}$$

This proves (ii). The final statement follows immediately from Prop.5.4. \square

6. \mathbb{Z}_3 -ORBIFOLD CONSTRUCTION AND THE MONSTER GROUP

In this section, we discuss an application of Corollary 4.8 and Proposition 5.17. The main purpose is to construct certain 3-local subgroups of the Monster simple group.

\mathbb{Z}_3 -orbifold of the Leech lattice VOA. Let Λ denote the Leech lattice and let τ be a fixed point free isometry of Λ . It is well-known [DLM00] that the lattice VOA V_Λ has exactly one irreducible τ^i -twisted module for $i = 1, 2$. We denote these twisted modules by $V_\Lambda^{T,1}$ and $V_\Lambda^{T,2}$, respectively. We define the $V_\Lambda[0]$ -module

$$V^\sharp := V_\Lambda[0] \oplus (V_\Lambda^{T,1})_{\mathbb{Z}} \oplus (V_\Lambda^{T,2})_{\mathbb{Z}},$$

where $(V_\Lambda^{T,i})_{\mathbb{Z}}$ is the subspace of $V_\Lambda^{T,i}$ of integral weights.

The next proposition is proved in [Miy13].

Proposition 6.1 ([Miy13]). *The module V^\sharp has a natural VOA structure. Moreover, it is a \mathbb{Z}_3 simple current extension of the VOA V_Λ^τ .*

Remark 6.2. That V^\sharp has a natural VOA structure was first announced by Dong and Mason [DM94]. They also claimed that the full automorphism group of V^\sharp is isomorphic to the Monster and $V^\sharp \cong V^\natural$ as a VOA. However, the complete proof has not been published.

In [SS10], a 3-local characterization of the Monster simple group has been obtained.

Theorem 6.3. [SS10, Thm. 1.1] *Let G be a finite group and $S \in \text{Syl}_3(G)$. Let H_1 and H_2 be subgroups of G containing S such that*

- M1:** $H_1 = N_G(Z(O_3(H_1)))$, $O_3(H_1)$ is extraspecial group of order 3^{13} , $H_1/O_3(H_1) \cong 2 \text{ Suz} : 2$ and $\text{Cent}_{H_1}(O_3(H_1)) = Z(O_3(H_1))$.
- M2:** $H_2/O_3(H_2) \cong \Omega_8^-(3)$, $O_3(H_2)$ is elementary abelian of order 3^8 and $O_3(H_2)$ is a natural $H_2/O_3(H_2)$ -module.
- M3:** $(H_1 \cap H_2)/O_3(H_2)$ is an extension of an elementary abelian group of order 3^6 by $2 \cdot \text{PSU}_4(3) : 2^2$.

Then G is isomorphic to the largest sporadic simple group, the Monster.

In this section, we will construct explicitly certain subgroups H_1 and H_2 of $\text{Aut}(V^\sharp)$ such that H_1 and H_2 satisfy the hypotheses [M1] to [M3] above.

Simple Current extension. Let $V(0)$ be a simple rational C_2 -cofinite VOA of CFT type and let $\{V(\alpha) \mid \alpha \in A\}$ be a set of inequivalent irreducible $V(0)$ -modules indexed by an abelian group A . A simple VOA $V = \bigoplus_{\alpha \in A} V(\alpha)$ is called a *A-graded extension* of $V(0)$ if $V(0)$ is a full sub VOA of V and V carries a A -grading, i.e., $Y(x^\alpha, z)x^\beta \in V^{\alpha+\beta}$ for any $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$. In this case, the group A^* of all irreducible characters of A acts naturally on V : for $\chi \in A^*$, $\chi(v) = \chi(\alpha)v$, $v \in V(\alpha)$. In other words, $V(\alpha)$ is an eigenspace of A^* for all $\alpha \in A$ and $V(0)$ is the fixed points of A^* .

If all V^α , $\alpha \in A$, are simple current $V(0)$ -modules, then V is referred to as a *A-graded simple current extension* of $V(0)$. The abelian group A is automatically finite since $V(0)$ is rational.

Theorem 6.4 ([Shi07, SY03]). *Let $V = \bigoplus_{\alpha \in A} V(\alpha)$ and $V' = \bigoplus_{\alpha \in A} V'(\alpha)$ be simple VOAs graded by a finite abelian group A . Suppose that $V(0) = V'(0)$ is a simple rational C_2 -cofinite VOA of CFT type and $V(\alpha)$ and $V'(\alpha)$ are simple current $V(0)$ -modules for all $\alpha \in A$. Let g be an automorphism of $V(0)$ which maps the set of isomorphism classes of $\{V(\alpha) \mid \alpha \in A\}$ to those of $\{V'(\alpha) \mid \alpha \in A\}$. Then there exists an isomorphism \tilde{g} from V' to V such that $\tilde{g}|_{V(0)} = g$.*

Notation 6.5. Let S_A be the set of the isomorphism classes of the irreducible $V(0)$ -modules $V(\alpha)$, ($\alpha \in A$). For an automorphism g of $V(0)$, we set $S_A \circ g = \{W \circ g \mid W \in S_A\}$, where $W \circ g$ denotes the g -conjugate of W , i.e., $W \circ g = W$ as a vector space and the vertex operator $Y_{W \circ g}(u, z) = Y_W(gu, z)$, for $u \in V$.

Define

$$H_A^N = \{h \in \text{Aut}(V(0)) \mid S_A \circ h = S_A\},$$

$$H_A^C = \{h \in \text{Aut}(V(0)) \mid W \circ h = W \text{ for all } W \in S_A\}.$$

Then we obtain the restriction homomorphisms

$$\begin{aligned}\Phi_A^N &: N_{\text{Aut}(V)}(A^*) \rightarrow H_A^N, \\ \Phi_A^C &: C_{\text{Aut}(V)}(A^*) \rightarrow H_A^C,\end{aligned}$$

Applying Theorem to the case $V = V'$, we show that Φ_A^N is surjective. Since each $V(\alpha)$ is irreducible, $\text{Ker } \Phi_A^N = A^*$. By similar arguments, Φ_A^C is surjective, and $\text{Ker } \Phi_A^C = A^*$.

Theorem 6.6 ([Shi07] (cf. [SY03])). *Let $V = \bigoplus_{\alpha \in A} V(\alpha)$ be a simple VOA graded by a finite abelian group A . Suppose that the fusion rule $V(\alpha) \times V(\beta) = V(\alpha + \beta)$ holds for all $\alpha, \beta \in A^*$. Then the restriction homomorphism Φ_A^N and Φ_A^C are surjective and $\text{Ker } \Phi_A^C = \text{Ker } \Phi_A^N = A^*$. That is, we have short exact sequences*

$$\begin{aligned}0 \longrightarrow A^* \longrightarrow N_{\text{Aut}(V)}(A^*) \longrightarrow H_A^N \longrightarrow 0, \\ 0 \longrightarrow A^* \longrightarrow C_{\text{Aut}(V)}(A^*) \longrightarrow H_A^C \longrightarrow 0,\end{aligned}$$

A subgroup of the shape $3^{1+12}(2\text{Suz} : 2)$. Next we will construct a subgroup of the shape $3^{1+12}(2\text{Suz} : 2)$ in $\text{Aut}(V^\sharp)$. First we recall a theorem from [LY13].

Proposition 6.7 ([LY13, Thm. 5.15]). *Let L be an even positive definite rootless lattice. Let ν be a fixed point free isometry of L of prime order p and $\hat{\nu}$ a lift of ν in $O(\hat{L})$. Then we have an exact sequence*

$$1 \longrightarrow \text{Hom}(L/(1-\nu)L, \mathbb{Z}_p) \longrightarrow \text{Cent}_{\text{Aut}(V_L)}(\hat{\nu}) \xrightarrow{\varphi} \text{Cent}_{O(L)}(\nu) \longrightarrow 1. \quad (6-1)$$

Recall that

$$V^\sharp = V_\Lambda[0] \oplus (V_\Lambda^{T,1})_{\mathbb{Z}} \oplus (V_\Lambda^{T,2})_{\mathbb{Z}}.$$

There is a natural automorphism τ' of order 3 which acts on V^\sharp as 1 on $V_\Lambda[0]$, as ξ on $(V_\Lambda^{T,1})_{\mathbb{Z}}$, and as ξ^2 on $(V_\Lambda^{T,2})_{\mathbb{Z}}$.

Proposition 6.8. *Let τ' be defined as above. Then*

$$N_{\text{Aut}(V^\sharp)}(\langle \tau' \rangle) \cong 3^{1+12} \cdot (2\text{Suz} : 2).$$

Proof. Let $S_A := \{V_\Lambda[0], (V_\Lambda^{T,1})_{\mathbb{Z}}, (V_\Lambda^{T,2})_{\mathbb{Z}}\}$ and

$$H_A^N = \{h \in \text{Aut } V_\Lambda[0] \mid S_A \circ h = S_A\}.$$

Recall that $V_\Lambda[0]$ has exactly nine irreducible modules and only $(V_\Lambda^{T,1})_{\mathbb{Z}}$ and $(V_\Lambda^{T,2})_{\mathbb{Z}}$ have top weight 2 (see [Miy13]). Therefore, S_A is invariant under the action of $\text{Aut } V_\Lambda[0]$ and $H_A^N = \text{Aut } V_\Lambda[0]$.

Now consider the simple current extension:

$$V_\Lambda = V_\Lambda[0] \oplus V_\Lambda[1] \oplus V_\Lambda[2],$$

which is graded by \mathbb{Z}_3 .

Let $S_B := \{V_\Lambda[0], V_\Lambda[1], V_\Lambda[2]\}$. Then we have a short exact sequence

$$0 \rightarrow \langle \hat{\tau} \rangle \rightarrow N_{\text{Aut}(V_\Lambda)}(\langle \hat{\tau} \rangle) \xrightarrow{\varphi} N_B^N \rightarrow 0,$$

where

$$H_B^N := \{h \in \text{Aut}(V_\Lambda^\tau) \mid S_B \circ h = S_B\}.$$

Since $V_\Lambda[1]$ and $V_\Lambda[2]$ are the only irreducible modules of $V_\Lambda[0]$ which have the top weight 1, all elements of $\text{Aut}(V_\Lambda^\tau)$ lift to $\text{Aut}(V_\Lambda)$ and hence

$$H_B^N = \text{Aut}(V_\Lambda^\tau) \cong N_{\text{Aut}(V_\Lambda)}(\langle \hat{\tau} \rangle) / \langle \hat{\tau} \rangle \cong 3^{12} \cdot N_{O(\Lambda)}(\tau) / \langle \tau \rangle.$$

Finally we will show that the subgroup $\varphi^{-1}(3^{12})$ is extra-special of order 3^{13} .

Let $V_\Lambda^{T,i} = S[\tau] \otimes T_i$ be the irreducible τ^i -twisted module of V_Λ for $i = 1, 2$. Recall the central extension (see [DL96])

$$1 \longrightarrow \langle -\xi \rangle \longrightarrow \hat{\Lambda}_{\tau^i} \twoheadrightarrow \Lambda \longrightarrow 1$$

associated to the commutator map $c^{\tau^i}(\alpha, \beta) = \xi^{\langle \alpha, \beta \rangle - \langle \tau^i \alpha, \beta \rangle}$, where $\xi^3 = 1$ and $i = 1, 2$.

Set $K = \{a^{-1}\hat{\tau}^i(a) \mid a \in \hat{\Lambda}_{\tau^i}\}$. Then $\hat{\Lambda}_{\tau^i}/K \cong 2 \times 3^{1+12}$ and $\hat{\Lambda}_{\tau^i}/K$ acts faithfully on T_i . By [FLM88, Prop. 5.5.3], there is an exact sequence

$$1 \rightarrow \mathbb{C}^\times \rightarrow N_{\text{Aut}(T_i)}(\pi(\hat{\Lambda}_{\tau^i}/K)) \xrightarrow{\text{int}} \text{Aut}(\hat{\Lambda}_{\tau^i}) \rightarrow 1,$$

where π is the representation of $\hat{\Lambda}_{\tau^i}/K$ on T_i and $\text{int}(g)(x) = gxg^{-1}$.

Consider a subgroup

$$X = \{(g, g', g'') \in \hat{\Lambda}/K \times \hat{\Lambda}_{\tau^1}/K \times \hat{\Lambda}_{\tau^2}/K \mid \varphi(g) = \text{int}(g') = \text{int}(g'')\}.$$

Then $X \cong 2 \times 3^{1+12}$. By the similar argument as in [FLM88, Section 10.4], one can show that X acts on the space $V_\Lambda \oplus V_\Lambda^{T,1} \oplus V_\Lambda^{T,2}$. Moreover, there is a canonical embedding $\nu : X \rightarrow N_{\text{Aut}(V^\#)}(\langle \tau' \rangle)$ such that $\varphi(\nu(3^{1+12})) = \text{Hom}(\Lambda/(1-\tau)\Lambda, \mathbb{Z}_3)$. \square

A subgroup of the shape $3^8 \cdot \Omega^-(8, 3)$. We will construct a subgroup H_2 of $\text{Aut}(V^\#)$ satisfying the following conditions:

- (i) $H_2/O_3(H_2) \cong \Omega_8^-(3) : 2$.
- (ii) $O_3(H_2)$ is elementary abelian of order 3^8 .
- (iii) $O_3(H_2)$ is a natural $H_2/O_3(H_2)$ -module.

Recall that the Coxeter-Todd lattice K_{12} can be constructed by using the Hexacode.

Let $\pi : \mathbb{Z}[\xi] \rightarrow \mathbb{Z}[\xi]/2\mathbb{Z}[\xi] \cong \mathbb{F}_4$ be the natural quotient, where ξ is the primitive cubic root of unity. We know the lattice K_{12} can be defined as

$$K_{12} := \{(x_1, \dots, x_6) \in (\mathbb{Z}[\xi])^6 \mid (\pi x_1, \dots, \pi x_6) \in \mathcal{H}\},$$

where \mathcal{H} is the hexacode.

Since \mathcal{H} is self-dual, every irreducible $V_{K_{12}}^\tau$ -module is a simple current module. There are exactly $3^6 \cdot 3 \cdot 3 = 3^8$ of them. These irreducible modules form an abelian group isomorphic to \mathbb{Z}_3^8 under the fusion product. Denote $R(V_{K_{12}}^\tau)$ the set of irreducible $V_{K_{12}}^\tau$ -modules.

As before, we denote irreducible $V_{K_{12}}^\tau$ -modules as

$$S^a[x], T^a[x], \text{ and } \check{T}^a[x]$$

for $a \in K_{12}^* \bmod K_{12}, x \in \mathbb{Z}_3$. If $a = \mathbf{0}$, we will omit this superscript. The fusion rules are given as follows:

$$S^a[x] + S^b[y] = S^{a+b}[x+y], \quad S^a[x] + T^b[y] = T^{b-a}[x+y], \quad (6-2)$$

$$S^a[x] + \check{T}^b[y] = \check{T}^{a+b}[y-x], \quad T^a[x] + \check{T}^b[y] = S^{b-a}[x-y], \quad (6-3)$$

$$T^a[x] + T^b[y] = \check{T}^{-(a+b)}[-(x+y)], \quad \check{T}^a[x] + \check{T}^b[y] = T^{-(a+b)}[-(x+y)]. \quad (6-4)$$

Note that the operation $+$ on the left hand side of the above equations is the fusion product.

Recall that $K_{12}^*/K_{12} \cong (\mathcal{H} \times \mathbf{0})^*/(\mathcal{H} \times \mathbf{0}) = 0 \times \mathbb{F}_3^6$ as an abelian group. It also forms a non-singular quadratic space of minus type (see for example [CS83]) if we define the quadratic form

$$q_F(a + K_{12}) = 3\langle a, a \rangle \bmod 3.$$

Proposition 6.9. *We define a map $q : R(V_{K_{12}}^\tau) \rightarrow \mathbb{Z}_3$ by*

$$q(S^a[x]) = q_F(a), \quad \text{and} \quad q(T^a[x]) = q(\check{T}^a[x]) = q_F(a) + x + 1.$$

Then q is a quadratic form on $R(V_{K_{12}}^\tau)$ and $(R(V_{K_{12}}^\tau), q)$ is non-singular space of minus type.

Proof. Let $B(x, y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$. Then B is a symmetric form and we have

- (i) $B(S^a[x], S^b[y]) = 3\langle a, b \rangle$;
- (ii) $B(S^a[x], T^b[y]) = 6\langle a, b \rangle + 2x$ and $B(S^a[x], \check{T}^b[y]) = 3\langle a, b \rangle + x$;
- (iii) $B(T^a[x], T^b[y]) = B(\check{T}^a[x], \check{T}^b[y]) = 3\langle a, b \rangle + 2(x+y-1)$;
- (iv) $B(\check{T}^a[x], T^b[y]) = B(T^a[x], \check{T}^b[y]) = 6\langle a, b \rangle + x+y-1$.

It is straightforward to check that this form is bilinear with respect to the fusion products and hence q is a quadratic form.

Using the bilinear form, it is clear that

$$R(V_{K_{12}}^\tau) = \{S^a[0] \mid a \in K_{12}^* \bmod K_{12}\} \perp \text{Span}\{S[1], T[1]\}.$$

Note that $\{S[1], T[1]\}$ is a hyperbolic pair and $\{S^a[0] \mid a \in K_{12}^* \bmod K_{12}\}$ is a quadratic space isometric to $K_{12}^*/K_{12} \cong \mathbb{F}_3^6$, which is a non-singular space of minus type. This completes our proof. \square

Lemma 6.10. *This quadratic form is $\text{Aut } V_{K_{12}}^\tau$ -invariant. That is for any $M \in R(V_{K_{12}}^\tau)$ and $g \in \text{Aut } V_{K_{12}}^\tau$, we have $q(M) = q(M \circ g)$.*

Proof. Recall that the weights of irreducible V_L^τ -modules are given by (see Tanabe and Yamada[TY07, (5,10)]):

$$\begin{aligned} \text{wt } V_{L^{0,j}}[\varepsilon] &\in \frac{2j^2}{3} + \mathbb{Z}, \\ \text{wt } V_L^{T,j}(\tau^i)[\varepsilon] &\in \frac{10 - 3(j^2 + \varepsilon)}{9} + \mathbb{Z}, \end{aligned}$$

for $i = 1, 2, j, \varepsilon \in \mathbb{Z}_3$.

Therefore, by the decompositions given in Prop. 3.7, we know that the weights of irreducible $V_{K_{12}}^\tau$ -modules are

$$\begin{aligned} \text{wt } S^a[\varepsilon] &\equiv \frac{2\langle a, a \rangle}{3} = \frac{2q(S^a[\varepsilon])}{3} \pmod{\mathbb{Z}}, \\ \text{wt } T^a[\varepsilon] = \text{wt } \check{T}^a[\varepsilon] &\equiv \frac{2(1 + \varepsilon + \langle a, a \rangle)}{3} = \frac{2q(T^a[\varepsilon])}{3} \pmod{\mathbb{Z}}. \end{aligned}$$

Since the action $M \mapsto M \circ g$ preserves weights, we are done. \square

Proposition 6.11. *Define a map $\phi : \text{Aut } V_{K_{12}}^\tau \rightarrow O(R(V_{K_{12}}^\tau), q)$ by*

$$g \mapsto (M \mapsto M \circ g).$$

Then ϕ is a group monomorphism and $\text{Im } \phi = {}^+\Omega_8^-(3) \cong \Omega_8^-(3).2$.

Proof. That ϕ is a group homomorphism follows from that $Y_{M \circ g}(u, z) = Y_M(gu, z)$.

We now prove that ϕ is injective. Let $g \in \text{Ker } \phi$ and consider the simple current extension:

$$V_{K_{12}} = V_{K_{12}}^\tau \oplus V_{K_{12}}[1] \oplus V_{K_{12}}[2].$$

Since $M \circ g \cong M$ for $M \in R(V_{K_{12}}^\tau)$, the set $S_B = \{V_{K_{12}}^\tau, V_{K_{12}}[1], V_{K_{12}}[2]\}$ is also fixed by g pointwise. Therefore, g can be lifted to $\text{Cent}_{\text{Aut}(V_{K_{12}})}(\langle \hat{\tau} \rangle)$. By Prop. 6.7, we have the exact sequence

$$0 \rightarrow \text{Hom}(K_{12}/(1 - \tau)K_{12}) \rightarrow \text{Cent}_{\text{Aut}(V_{K_{12}})}(\hat{\tau}) \rightarrow \text{Cent}_{O(K_{12})}(\tau) \rightarrow 0.$$

Recall that $O(K_{12}) \cong 6.\text{PSU}_4(3).2^2 \cong 3.O_6^-(3)$ (see for example [CS83]) and $\text{Cent}_{O(K_{12})}(\tau) \cong 6.\text{PSU}_4(3).2$. Therefore, $\text{Cent}_{O(K_{12})}(\tau)/\langle \tau \rangle \cong 2.\text{PSU}_4(3).2 \cong \Omega_6^-(3)$.

By direct calculations, it is easy to verify that $\text{Cent}_{\text{Aut}(V_{K_{12}})}(\hat{\tau})/\langle \hat{\tau} \rangle$ acts faithfully on $R(V_{K_{12}}^\tau)$. Hence $\ker \phi = \text{id}$.

Next we determine the image of ϕ . In [LY13], a subgroup $H \cong {}^+\Omega^-(8, 3)$ is constructed explicitly using σ -involutions associated to $c = 4/5$ Virasoro vectors (see [LY13, Rmk 5.52, Thm 5.64]). Since ${}^+\Omega^-(8, 3)$ is an index 2 subgroup of the full orthogonal group $O_8^-(3)$, we have $\text{Im } \phi \cong {}^+\Omega^-(8, 3)$ or $O_8^-(3)$.

Suppose $\text{Im } \phi \cong O_8^-(3)$. In this case, $Z(\text{Aut}(V_{K_{12}}^\tau)) \cong \mathbb{Z}_2$.

Let h be an order 2 element in $Z(\text{Aut}(V_{K_{12}}^\tau))$. Then $\phi(h)$ is the -1 map on $R(V_{K_{12}}^\tau)$. By the fusion rules (see Equation 6-2), we have

$$V_{K_{12}}[1] \circ h = \phi(h)(V_{K_{12}}[1]) \cong V_{K_{12}}[2] \quad \text{and} \quad V_{K_{12}}[2] \circ h = \phi(h)(V_{K_{12}}[2]) \cong V_{K_{12}}[1].$$

Therefore, h lifts to $\text{Aut}(V_{K_{12}})$ and is contained in the subgroup isomorphic to

$$N_{\text{Aut } V_{K_{12}}}(\hat{\tau})/\langle \hat{\tau} \rangle \cong 3^6 : (2.\text{PSU}_4(3).2^2),$$

which is centerless. It is a contradiction and hence $\text{Im } \phi \cong {}^+\Omega^-(8, 3) \cong \Omega_8^-(3).2$. \square

For now on, we denote $R(V_{K_{12}}^\tau)$ by R for simplicity. Notice that $(R, -q)$ also forms a non-singular quadratic space of minus type. Therefore, $(R, -q) \cong (R, q)$.

Let $\eta : (R, -q) \rightarrow (R, q)$ be a linear isometry and set

$$S_\eta = \{(a, \eta(a)) \in R \times R \mid a \in R\}.$$

Lemma 6.12. *The set S_η is a maximal totally singular subspace of $R \times R$. Moreover, the minimal conformal weight of S_η is 2.*

Proof. It is clear that S_η is a vector subspace of $R \times R$ and $\dim_{\mathbb{F}_3} S_\eta = \dim_{\mathbb{F}_3} R = 8$.

By the definition of η , we also have

$$q(a, \eta(a)) = q(a) + q(\eta(a)) = q(a) - q(a) = 0 \quad \text{for all } a \in R$$

Therefore, S_η is totally singular. It is maximal since $\dim_{\mathbb{F}_3} S_\eta = 1/2 \dim_{\mathbb{F}_3}(R \times R)$.

Recall that the conformal weights of the elements in R are given by

$$\text{wt}(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } q(a) = 0, a \neq 0, \\ 4/3 & \text{if } q(a) = 1, \\ 2/3 & \text{if } q(a) = 2. \end{cases}$$

Therefore, $\text{wt}(a, \eta(a)) = 2$ if $a \neq 0$. \square

Lemma 6.13. *Let S be a maximal totally singular subspace of $R \times R$ such that the minimal conformal weight of S is ≥ 2 . Then there is a linear isomorphism $\eta : R \rightarrow R$ such that $q(\eta(a)) = -q(a)$ for all $a \in R$ and $S = S_\eta$.*

Proof. Let $p_i : R \times R \rightarrow R, i = 1, 2$, be the natural projection to the i -th coordinate.

Let $(a, b) \in S$ be a non-zero vector. Then neither a nor b is zero; otherwise, $\text{wt}(a, b) = \text{wt}(a) + \text{wt}(b) = 1$. That means $p_i|_S$ is injective for any $i = 1, 2$ and hence $p_i|_S$ is bijective for any $i = 1, 2$ since $\dim_{\mathbb{F}_3}(S) = \dim_{\mathbb{F}_3}(R)$.

Let $\eta = p_2 \circ (p_1|_S)^{-1}$. Then $\eta : R \rightarrow R$ is a linear isomorphism and

$$S = \{(a, \eta(a)) \mid a \in R\}.$$

Since S is totally singular, we have $q(a, \eta(a)) = q(a) + q(\eta(a)) = 0$ for any $a \in R$ and hence $q(\eta(a)) = -q(a)$ for all $a \in R$. \square

Proposition 6.14. *Let \mathcal{S} be the set of all maximal totally singular subspaces of $R \times R$ such that the minimal conformal weight is ≥ 2 . Then \mathcal{S} is transitive under the action of $O_8^-(3) \wr 2$.*

Proof. This follows immediately from Lemmas 6.12 and 6.13. \square

Lemma 6.15. *Let S^\sharp be a maximal totally isomorphic subspaces of $R \times R$ such that $\oplus_{M \in S} M = V^\sharp$. Then*

$$\text{Stab}_{\text{Aut}(V_{K_{12}}^\tau \otimes V_{K_{12}}^\tau)}(S^\sharp) \cong \text{Aut}(V_{K_{12}}^\tau) \cong \Omega_8^-(3).2.$$

Proof. By Lemma 6.13, $S^\sharp = \{(a, \eta(a)) \mid a \in R\}$ for some linear isomorphism $\eta : R \rightarrow R$ such that $q(\eta(a)) = -q(a)$.

Note also that for any $g \in O(R, q)$, $\eta g \eta^{-1}$ is also in $O(R, q)$. Moreover, the map

$$\begin{aligned} \xi : O(R, q) &\rightarrow O(R, q) \times O(R, q) \\ g &\mapsto (g, \eta g \eta^{-1}) \end{aligned}$$

is a group monomorphism. It is also easy to verify that $\text{Stab}_{O(R, q) \wr 2}(S^\sharp) = \xi(O(R, q))$. Hence, we have the desired conclusion. \square

Proposition 6.16. *Let S^\sharp be defined as in Lemma 6.15. Let A be the abelian subgroup of $\text{Aut}(V^\sharp)$, which acts on V^\sharp as the dual group of S^\sharp . Then*

$$N_{\text{Aut}(V^\sharp)}(A) \cong 3^8.\Omega_8^-(3).2.$$

Proof. This follows from Theorem 6.6 and Lemma 6.15. \square

We have constructed two subgroups

$$\begin{aligned} H_1 &= \text{Stab}_{\text{Aut}(V^\sharp)}(V_\Lambda^\tau) = N_{\text{Aut}(V^\sharp)}(\langle \tau' \rangle) \cong 3^{1+12}.(2.\text{Suz} : 2), \text{ and} \\ H_2 &= \text{Stab}_{\text{Aut}(V^\sharp)}(V_{K_{12}}^\tau \otimes V_{K_{12}}^\tau) = N_{\text{Aut}(V^\sharp)}(3^8) \cong 3^8.\Omega_8^-(3).2. \end{aligned}$$

Lemma 6.17. *The intersection of H_1 and H_2 is the common stabilizer of the subVOAs V_Λ^τ and $V_{K_{12}}^\tau \otimes V_{K_{12}}^\tau$ and $H_1 \cap H_2 = 3^8.(3^6.(2\text{PSU}_4(3).2^2))$.*

Proof. Recall the exact sequence

$$1 \rightarrow 3^8 \rightarrow H_2 \rightarrow \Omega_8^-(3).2.$$

By definition, it is clear that the normal subgroup 3^8 stabilizes all irreducible $V_{K_{12}}^\tau \otimes V_{K_{12}}^\tau$ submodules in S^\sharp and hence it stabilizes V_Λ^τ , also.

Note that H_2 acts on (R, q) as a subgroup of isometries. The subgroup that stabilizes V_Λ^τ will stabilize a 6-dimensional non-singular subspace of R and it has the shape

$$3^6 : (2.\text{PSU}_4(3).2^2) \quad (\text{see } [\text{CCN}^+85, \text{page } 141]).$$

Hence, $H_1 \cap H_2 \cong 3^8.(3^6 : (2.PSU_4(3).2^2))$. \square

Remark 6.18. Unfortunately, we do not have a direct proof that $\text{Aut}(V^\sharp)$ is finite and hence we cannot apply Theorem 6.3 to conclude that $\text{Aut}(V^\sharp)$ is isomorphic to the Monster.

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